THE THEORY OF ELECTROSTATIC PROBES
IN STRONG MAGNETIC FIELDS

by

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A theory is developed of an electrostatic probe in a fully-ionized plasma in the presence of a strong magnetic field. The ratio of electron Larmor radius to probe transverse dimension is assumed to be small. Poisson's equation, together with kinetic equations for ions and electrons are considered. An asymptotic perturbation method of multiple scales is used by considering the characteristic lengths appearing in the problem. The leading behavior of the solution is found. The results obtained appear to apply to weaker fields also, agreeing with the solutions known in the limit of no magnetic field. The range of potentials for which results are presented is limited.

The basic effects produced by the field are a depletion of the plasma near the probe and a non-monotonic potential surrounding the probe. The ion saturation current is not changed but changes appear in both the floating potential V_f and the slope of the current-voltage diagram at V_f. The transition region extends beyond the space potential V_s, at which point the current is largely reduced.

The diagram does not have an exponential form in this region as commonly assumed. There exists saturation in electron collection. The extent to which the plasma is disturbed is determined. A cylindrical probe has no solution because of a logarithmic singularity at infinity. Extensions of the theory are considered.

This abstract is approved as to form and content. I recommend its publication.

Signed ______________________________
Faculty member in charge of dissertation

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Chapter I

Introduction

A. The electrostatic probe

In his pioneer work on ionized gases, Langmuir [1] developed the electrostatic probe as a useful diagnostic tool. The probe is a small metallic electrode biased with respect to the electric potential of the surrounding plasma by means of external circuitry. The current it collects as a function of the voltage applied can provide information on the local state of the plasma (electron and ion density and temperatures and plasma potential). The local character of the measurement and its experimental simplicity are advantages not present in most other plasma diagnostic techniques. However, except for the special conditions for which it was first proposed, the theory is rather complicated.

The theory of electrostatic (or Langmuir) probes seeks to predict the size and shape of the probe characteristic or $I - V_p$ diagram (the relation between collected current and collecting voltage), and its dependence on the parameters N_∞, T_e, T_i and V_s. An experimental registration of the actual diagram then allows calculation of the values of the aforementioned plasma parameters. What makes the problem difficult and interesting is its boundary-layer character. In the body of the plasma quasineutrality exists but near boundaries, such as that of the probe, large differences in charge appear.

In the absence of an external magnetic field $\vec{B}$, the characteristic has the qualitative behavior of Fig. 1. V_s is the space potential (the local value of plasma potential). For $V_p \gg V_s$ only electrons are collected; moreover if the Debye length, λ_D, is much smaller than the probe radius, R, an electron saturation current I_s^e is reached. If in addition, $T_i \ll T_e$, the saturation is reached almost immediately beyond V_s. On the other hand if λ_D is comparable to R, no saturation appears.

The b-region is called the transition region; electrons are repelled by the probe but because their thermal velocity is much larger than that of ions the current is still negative. As V_p decreases in value, a potential V_f is reached for which the net current to the probe is zero. V_f is called the floating potential because it is identical to the terminal voltage of an

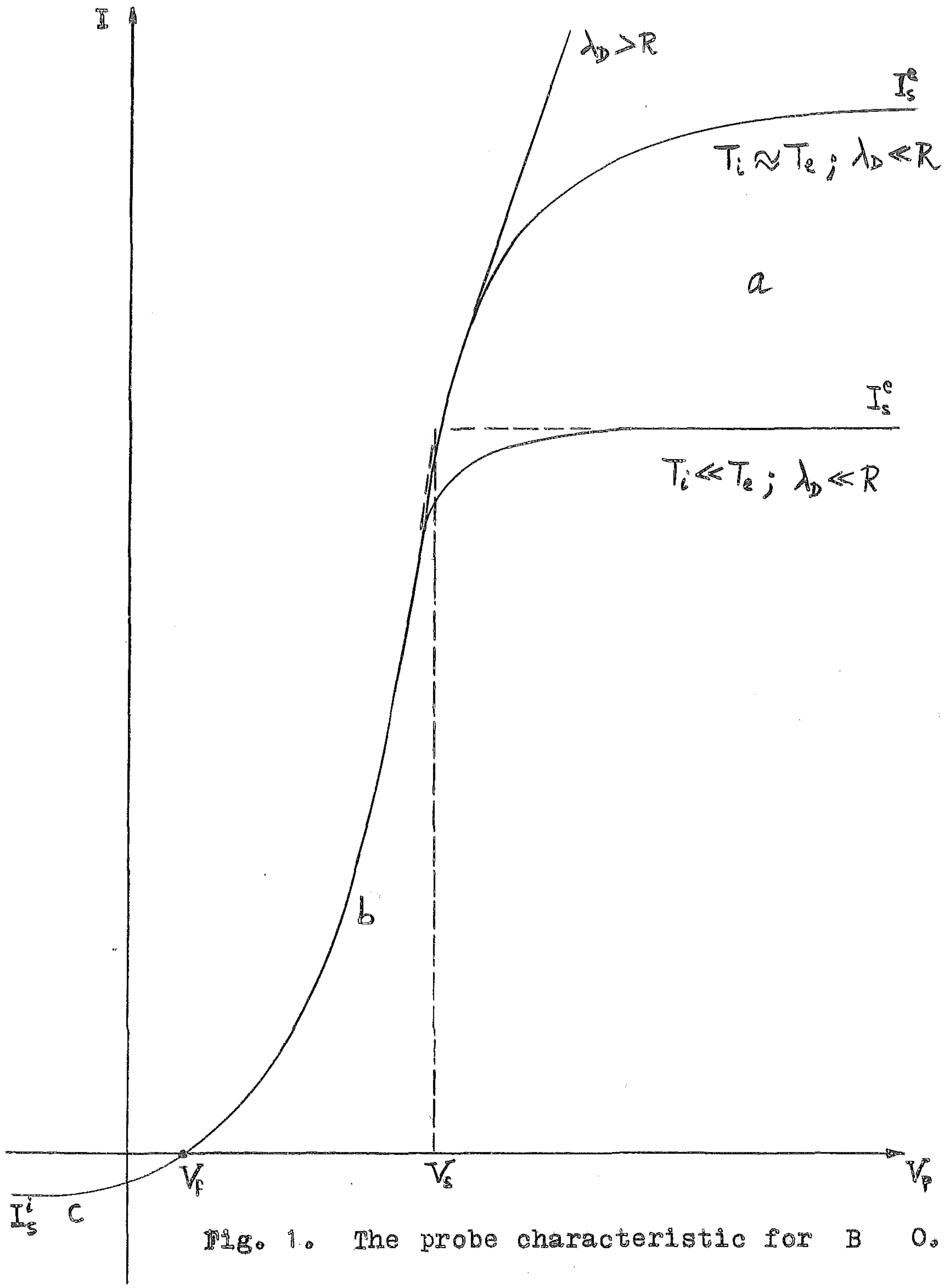

Fig. 1. The probe characteristic for B 0.

electrically isolated collector. Finally, for V_p sufficiently negative with respect to V_s, an ion saturation region is reached (c-region).

A rough idea of the relation of plasma parameters to the $I - V_p$ diagram features is as follows: I_s^e is proportional to $N_\infty \left(\frac{kT_e}{m_e}\right)^{1/2}$ (k = Boltzmann constant, m_e = electron mass); I_s^i is proportional to $N_\infty \left(\frac{kT_i}{m_i}\right)^{1/2}$; and in the transition region $\ell n I$ is linear in V_p, the slope being inversely proportional to T_e, if the electrons have a Maxwellian distribution. In other cases its form can give information in the distribution function itself. V_s is normally interpreted as the "bend" between a and b (if $\lambda_D \ll R$); it is explicitly determined by extrapolating parts a and b of the characteristic to an intersection.

When a magnetic field is present ($B \neq 0$), the $I - V_p$ diagram has less definite features; indeed they are difficult to define. The most relevant and confirmed are a decrease in I and a degradation of the saturation character of the electron current at large V_p (fig. 2).

In section B we discuss previous theories for both cases ($B = 0$, $B \neq 0$); in Section C we discuss the assumptions used to define clearly the problem which is mathematically formulated in Section D.

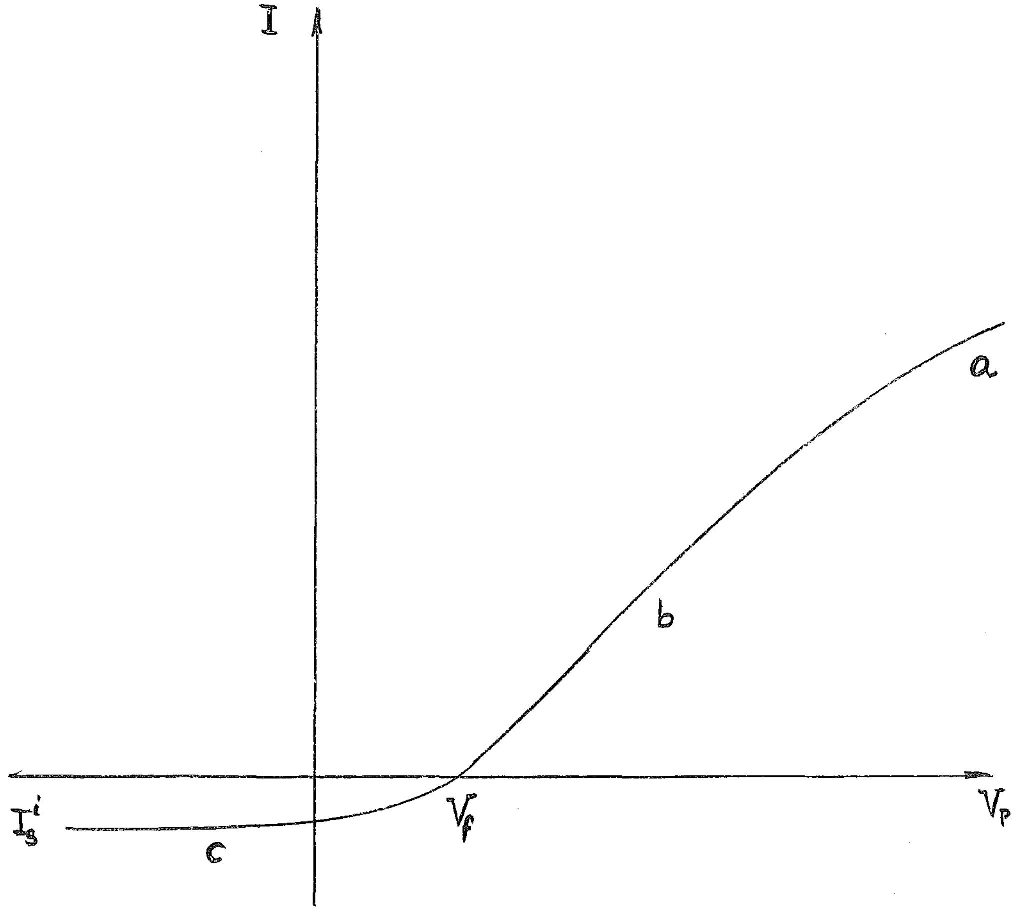

Fig. 2. The probe characteristic for large B.

Chapter II describes the mathematical method of attacking the problem: the kinetic equations are made nondimensional and the asymptotic expansion method is established.

In Chapter III we obtain and discuss the general characteristics of the solution. Chapter IV considers the construction of the I - V_p diagram and the numerical results are presented and discussed. Finally in Chapter V the extension of the method to other situations is considered and a resume of conclusions is given.

B. Survey of literature

B1. Theories for no magnetic field

Probe theory has been developed primarily for plasmas without magnetic fields. Plasmas without magnetic fields can be classified according to their plasma density (see 2 and Ch. II - Sect. B): dilute, continuum and dense plasmas. Continuum, fully-ionized plasmas are not possible.

The old theory of Langmuir is concerned with dilute plasmas. They are characterized by having both λ_D and R much smaller than the mean free path, λ. The treatment is then purely collisionless. Langmuir [3] demonstrated that in many cases the region in which ion and electron densities can differ is confined to a very thin "sheath", and quasineutrality exists outside it; moreover, the electric field is concentrated inside the sheath. This is the case if $\lambda_D \ll R$. He also proved that electron or ion collection saturates for large values of $|V_p|$.

When $\lambda_D \sim R$, no saturation appears [4]; in the limit $\lambda_D \gg R$, I grows linearly with V_p for a spherical probe and as $V_p^{1/2}$ for a cylinder. For most plasmas of interest, $\lambda_D \ll R$.

In 1949, Bohm [5] proved that Langmuir's theory is inadequate for ion collection when $T_i \ll T_e$ (in general for the collection of the colder species). This inequality is satisfied in many gas discharges in which ion collection must be used because the large electron currents seriously perturb the plasma. Moreover, when $T_i \ll T_e$ the simple orbital analysis of Langmuir is also inadequate because the electric field is not completely confined to the sheath and an absorption radius appears. In brief, Langmuir's original theory is correct for collection of the hotter species of a plasma whose temperature ratio $\frac{T_i}{T_e}$ is very far from unity. Then, saturation of the hotter species is reached almost immediately beyond V_s.

Bohm's criterion stated that for $T_i \ll T_e$ and ion collection, the ions had to enter the sheath with a drift velocity much larger than their thermal velocity; therefore, not all of the electric field imposed by the probe is confined to the sheath. As a result, the ion distribution function is unknown at the sheath edge.

Allen et al. [6] attacked this problem in the limit of $\frac{T_i}{T_e} \rightarrow 0$. Berstein and Rabinowitz [7] considered the case of finite ion temperature. In both works Poisson's equation had to be solved: the collected current depends on the potential field while in Langmuir's theory does not.

The method used in [7] was a solution of the Vlasov eq., giving the ion distribution function as a function of the constants of the motion. For highly symmetrical geometries (spherical and cylindrical probes) these constants are as many as the velocity coordinates. The density can be found by integration over the range of variation of these constants of the motion, eliminating explicit trajectory calculations. Results were given for monoenergetic ions, but are of difficult use.

Lam [8], using the same theory, made an asymptotic analysis in the limit $\frac{\lambda_D}{R} \rightarrow 0$ and monoenergetic ions. Bienkowski [9] presented some corrections.

At present, a complete treatment of the dilute plasma exists for the whole $I - V_p$ diagram, and arbitrary ratios of $\frac{\lambda_D}{R}$ and $\frac{T_i}{T_e}$. Laframboise [10] has very recently used the method of Bernstein and Rabinowitz to treat plasmas with Maxwellian distribution functions at infinity. These important calculations will prove useful in our problem (see Chapter IV, Sections A and B).

The probe theory of continuum plasmas is, while less complete, satisfactory. The two basic papers are by Su and Lam [11] and Cohen [12]. In a continuum plasma, $\lambda \ll \lambda_D$ and $\lambda \ll R$. In [11] the limit $\frac{\lambda_D}{R} \rightarrow 0$ was taken; moreover, either $\frac{T_e}{T_i}$ or the probe potential was allowed to go to infinity. Only the leading behavior was found.

Cohen also used the limit $\frac{\lambda_D}{R} \rightarrow 0$, but both the temperature ratio and probe potential were arbitrary. First order corrections were found.

In both studies the plasmas were weakly ionized (in a classical, fully ionized plasma $\lambda_D \ll \lambda$ always). Mobility and diffusion coefficients were then constants and transport macroscopic equations were used up to the probe surface.(*)

(*) Su and Sonin [13] extended the results to not so weakly ionized plasmas.

The intermediate type of plasma, which we term a "dense" plasma, satisfies the inequalities $\lambda_D \ll \lambda \ll R$. No satisfactory theory exists at present for this case or for the more general case characterized by the mean free path being neither the shortest nor the longest length relevant to the problem. Davydov and Zmanovskaja [14], Boyd [15], Ecker et al. [16] and Waymouth [17], have all attempted some kind of ad hoc matching between the collision-dominated and collision-free regions.

Attempts to use kinetic theory were made by Su et al. [2] and Chou et al. [18]. The first investigation by Su used the two-sided distribution function method of Lees [19]. In [18], an unsuccessful attempt was made to use the kinetic eqs. directly.

B2. Theories for non-zero magnetic fields

Contrary to the enormous body of literature in the aforementioned problems, very few papers have been published on probe theory when a magnetic field is present.

Three basic difficulties separate this problem from those discussed above. First, the presence of $\vec{B}$ introduces an anisotrophy in space, i.e. the situation is at least two-dimensional; partial differential equations substitute throughout for ordinary differential equations. Second, because only collisions produce diffusion across B, the problem, while not being collision-dominated (except for the uninteresting continuum case), is not collisionless. The essential difficulty arises in the appearance of various regions where equations of different character have to be used. Finally, for highly-ionized gases, the spatial variation in density produces spatial changes in the transport coefficients.

No previously-published theories have satisfactorily treated the electrostatic probe problem with a magnetic field. Spivak and Reichrudel [20], Bickerton [21] and Nobata [22] considered plane probes parallel to $\vec{B}$ and infinite in extent so as to avoid the two-dimensional difficulty. However, the balance of fluxes across and along the field is of fundamental importance as we shall see. Any real finite probe will differ in essential ways from an infinite one. Moreover, all these authors had to assume some ill-defined sheath edge where the density was certainly different from the value in the unperturbed plasma. There the distribution function was assumed and an integration (taking into account the orbiting electrons) was made to obtain the flux to the probe.

Two other approaches exist. Bertotti [23] considered collection along $\vec{B}$ and averages of the various magnitudes over the probe cross section. Thus he reduces the problem to a one-dimensional one. An unspecified anomalous diffusion process is introduced and in this way a phenomenological integro-differential equation is found and solved numerically. The results are in clear contradiction with all available experimental results.

Bohm [5] has developed the most satisfactory analysis of the present problem. He finds in agreement with experiments a decrease in the collected current whose order of magnitude will be confirmed in the present study. He observed also the depletion of the plasma near the probe and the insensibility of the results to the shape of the probe along $\vec{B}$.

Basically he established a balance between transverse and longitudinal fluxes using macroscopic equations from the beginning. However, a number of important defects are present. First, he assumed a probe potential such which does not affect the electrons at all while the ions are completely repelled. This requires $T_i \ll T_e$. Second, the result found does not depend on V_p so that one does not know to which value of V_p the theory corresponds. Third, the diffusion equation is assumed to be valid up to a vaguely-defined distance from the probe: one mean free path along $\vec{B}$ and one electron Larmor radius across $\vec{B}$. Fourth, the density is assumed constant on this imaginary surface. Moreover, he considered only weakly ionized gases. Sugawara [24] has published an erroneous correction to Bohm's result.

In closing this review, it should be pointed out that all previous treatments including magnetic fields have considered only isolated parts of the $I - V_p$ diagram. General reviews of probe theory are given in [25].

C. Statement and discussion of assumptions

C1. Introduction

Two shapes of probes will be considered simultaneously: a thin disc of radius R and a thin strip of width $2R$ and infinitely long; both in a plane perpendicular to $\vec{B}$. They give rise to axisymmetric and two-dimensional problems respectively. Normally a sphere and a cylinder are chosen for $B \neq 0$; here they do not present any advantages due to their high symmetry because the magnetic field does not allow solutions which depend only on the radial distance. As we shall see, however, most of the $I - V_p$ diagram is not sensitive to the probe dimension parallel to $\vec{B}$.

The plasma to be considered has a negligible concentration of neutrals, one ion species with charge $+Z_i e$, and is in a steady, quiescent state. It is a classical plasma in the sense that N_D, the number of particles in a Debye sphere is very large. Although N_D grows with T_e and is inversely proportional to $N_\infty^{1/2}$, it there is a limit for high temperatures because the thermal De Broglie wave length decreases as $T_e^{-\frac{1}{2}}$ while the classical distance of closest approach or Landau length, $\lambda_L = \frac{e^2}{kT_e}$, decreases as T_e^{-1}. As a rough limit we require $T_e < 10^5$ °K. This also allows the exclusion of relativistic effects. Finally, because of the low-temperature type of plasma, no radiative or inelastic interactions are considered.

The ions and electrons are then described by appropriate kinetic equations with pure Coulomb interaction under a strong, uniform $\vec{B}$. Maxwell's equations reduce to Poisson's equation because: a) the problem is steady; and b) currents produce a negligible additional magnetic field. This is consistent also with the low-temperature plasmas considered, but we shall check it in Ch. II.

The probe normally is cold with respect to the plasma, and it acts as a sink for all particles. We shall assume it to be perfectly absorbing.

C2. The kinetic equations

It remains to specify the equations satisfied by the electron and ion distribution functions F^e, F^i. A homogeneous, isolated gas with widely separated collision and relaxation scales has a Boltzmann-like eq. describing its approach to equilibrium in phase space:

$$\frac{\partial F}{\partial t} = \frac{\delta F}{\delta t} \qquad (1.1)$$

The second member of (1.1) is the collision operator acting on F; it is nonlinear and its form depends on the type of gas. For a classical plasma ($N_D \gg 1$) it is given by the Balescu-Lenard (B-L) collision operator [26], [27]. Recent improvements (*) by Hubbard [28], Frieman and Book [29], Weinstock [30], Kihara and Aono [31] and Guernsey [32] do not seem to alter any result significantly. In fact the older Fokker-Planck (F-P) equation also gives very similar results.

A simple analysis (see App. B) shows that for a plasma the mean free path, λ, can be expressed as

$$\lambda = \frac{b'}{N_\infty \lambda_L^2}$$

b' being a non-dimensional number. Since Chandrasekhar's [33] and Spitzer's [34] papers, it is known that b' is not a constant, and a more exact expression is

(*) All of them remove the divergence present in the B-L model for very short interaction distances.

$$\lambda = \frac{b}{N_\infty \lambda_L^2 \ln \Lambda}$$

$$\Lambda = \frac{3\lambda_D}{\lambda_L} = 9 N_D$$

where b is approximately $\frac{1}{2\pi}$ and may be as large as $\frac{1}{\pi}$ or as small as $\frac{1}{4\pi}$, depending upon the process considered. (One exception is the mean free path for energy interchange between ions and electrons, where $b \sim \left(\frac{m_i}{m_e}\right)^{1/2}$.)

This leads immediately to (see Appendix B)

$$\frac{\lambda_D}{\lambda_L} = \frac{\ln \Lambda}{b\, 4\pi/3\, \Lambda} \ll 1$$

The collision and relaxation times, which are found by dividing λ_D and λ respectively by an average velocity, are thus widely separated as required.

When inhomogeneities and external fields are present and they are weak enough, (1.1) can still be used. This is accomplished by adding a Lionville-type operator to $\frac{\partial F}{\partial t}$ which is simply the divergence of the particle flux in phase-space produced by these external perturbations in the absence of particle interactions. This means that both effects are added linearly. The complete equation is (*):

$$\frac{\partial F}{\partial t} + \vec{w} \cdot \frac{\partial F}{\partial \vec{r}} + \vec{a} \cdot \frac{\partial F}{\partial \vec{w}} = \frac{\delta F}{\delta t} \qquad (1.2)$$

where $\vec{r}, \vec{w}$ are vector coordinates of space and velocity and $\vec{a}$ is the acceleration caused by the external fields.

When fields or inhomogeneities are not weak the description of the problem is much more involved. The collision operator is neither local nor instantaneous and involves the fields. It should be emphasized that the criterion of "weakness" requires the collision time $\frac{\lambda_D}{U}$ (U being an average velocity) to be much smaller than τ, the characteristic time of the field; or $\lambda_D/K^{-1} \ll 1$, K being the wave-number of inhomogeneities. In writing an equation like (1.2) for a plasma, λ_D is the critical length for determining the applicability of the Boltzmann-like equation and the magnitudes of the ratios $\lambda/U\tau$, λ/K^{-1} do not matter as long as $\lambda_D \ll \lambda$. This is frequently overlooked. Of course, for the solution of (1.2) itself, the relative sizes of λ and K^{-1} are important. (See Severne [35]).

(*) When coordinates other than cartesian are used, care must be taken when passing from momentum to velocity coordinates. Appendix A develops the left hand side of (1.2) for such a case.

In the present problem, inhomogeneities and electric and magnetic fields exist. We state now:

a) Whenever the specific form of $\frac{\delta F}{\delta t}$ is considered as we proceed with the problem in Chapter III, it will be easily seen that $K^{-1} \gg \lambda_D$ so that the local gradients are in this sense weak.

b) Similarly for the electric field, $\vec{E}$, it can be demonstrated that

$$\tau^{-1} = \frac{e|\vec{E}|}{m_e U_e} \ll \left(\frac{\lambda_D}{U_e}\right)^{-1} \sim \omega_p \quad \text{(plasma frequency)}$$

or

$$\frac{eV}{L} \ll \frac{kT_e}{\lambda_D}, \quad \text{or} \quad |\vec{E}| \ll \left(4\pi N_\infty kT_e\right)^{1/2} \qquad (1.3)$$

where τ is a measure of the relative strength of the external electric field. (The momentum of the ions is much larger, in general, than that of electrons, and thus the electrons are more critical for thepresent criterion).

In general, for both a) and b) it can be observed that since $\lambda_D^{-1} \gg \lambda^{-1}$, and since $\frac{\delta F}{\delta t}$ is proportional to λ^{-1} when properly normalized, the collision operator is not an important term except when gradients in the local field are, inthis sense, very weak.

We conclude that, as far as inhomogeneities and the electric field are concerned, $\frac{\delta F}{\delta t}$ will not involve the use of the local B-L expression for the collision operator.

them. This justifies the use of the local B-L expression for the collision operator.

c) The magnetic field is much more critical in this problem. The electron Larmor radius, ℓ_e, characterizes the strength of $\vec{B}$, where

$$\ell_e = \frac{m_e v_e c}{eB}$$

If

$$\ell_e \gg \lambda_D$$

we reach the same conclusions as in a) and b) above. However, if the case

$$\ell_e \ll \lambda_D$$

is to be considered, a modified B-L operator has to be used. This will now be discussed in some detail.

First, throughout a large part of the development to be given in Chapter III only the magnitude of $\frac{\delta F}{\delta t}$ is required. As shown in Appendix B the B-L model does not modify the results for the F-P equation. We now show that under the condition $\ell_e \gg \lambda_L$ (*), and irrespective of the value of ℓ_e / λ_D, these results are not modified in order of magnitude by the presence of the magnetic field.

(*) For $B = 10^4$ Gauss, and $T_e \approx 2.3 \times 10^3$ °K, $\ell_e \approx 10^{-4}$ cm and $\lambda_L \approx 10^{-6}$ cm; because ℓ_e grows and λ_L decreases as T_e increases, a plasma rarely exists not satisfying this condition.

Belyaev [36], as far as the author knows, was the first to take the presence of a strong, external magnetic field into account in the collision operator. He used Bogoliubov's theory [37] and a weak interaction (F-P) approximation. A basic result, contradicting an older analysis by Lifshitz, was that no change of order of magnitude appeared in the relaxation times. Belyaev and also Silin [38] suggested the substitution of ℓ_e for λ_D as screening distance when $\ell_e \ll \lambda_D$. Rostoker and Rosenbluth [39] again made clear that if $\lambda_D \ll \ell_e$ (and $K^{-1} \gg \ell_e$), $\vec{B}$ does not modify the collision operator. If $\ell_e \ll \lambda_D$ the screening distance seemed to be between ℓ_e and λ_D. Finally Rostoker [40], Sundaresan and Wu [41], and Haggerty and Sobrino [42] derived a modified B-L collision operator which included a magnetic field.

It was shown in [41] and [42] that, for $B \to \infty$, no interchange of energy is possible between motions along and across $\vec{B}$ (therefore the relaxation times for some processes appear to become infinite).

No contradiction exists between [36] and these results. As seen in [43] (see Ch. 9, Eq. 9.29c) the limit $B \to \infty$ is handled by letting the variable

$$N \equiv \frac{K_\perp P_\perp}{m\,\omega}$$

vanish ($\frac{P_\perp}{m\omega} = \frac{V_\perp}{\omega}$ is the Larmor radius of a particle and $K_\perp$ is the transverse wave number of interaction). But if $\ell_e \gg \lambda_L$, as considered by Belyaev realistically, there is a long range of $K_\perp$ for which N is not small, i.e., for which

$$\lambda_L \ll K_\perp^{-1} \lesssim \ell_e$$

even if $\ell_e \ll \lambda_D$. In fact a Coulomb interaction with a cutoff at λ_L has an effective collision length only $\ln\Lambda$ longer than a Coulomb interaction cut at λ_D, $\ln\Lambda$ being relatively large ($\ln\Lambda = O(10)$). For $\ell_e \ll \lambda_D$ but $\ell_e \gg \lambda_L$, the factor $\ln\Lambda(\ell_e)$ would be between 1 and $\ln\Lambda(\lambda_D)$, the order of magnitude of λ changing by a factor of 2 or perhaps 3.

Second, when an explicit form of the collision term has to be used, the influence of $\vec{B}$ has to and will be considered. This is done in Appendix E.

C3. The steady character of the problem

Now we make clear a final assumption; this is that we can write in (1.2)

$$\frac{\partial F}{\partial t} = 0$$

While this is legitimate because the plasma is assumed to be in a steady state and the measurement is made in scales of time clearly much larger than any process of interest in the plasma (even for a fast-sweeping voltage source (*)), it implies the existence of a steady state.

There are two possible ways in which this is not true; first, if the electric field is large enough to produce runaway effects [44]. It is known that this condition is much more critical than (1.3); it can be expressed as

$$|\vec{E}| \ll \frac{2\pi N_\infty e^3}{kT_e \ln N_D}$$

The strongest gradients will appear perpendicular to $\vec{B}$, but $\vec{B}$ inhibits any runaway of electrons. Along $\vec{B}$ the field will be very weak except in a relatively short region. An acceleration is needed over longer distances to produce runaway electrons and this does not seem to happen here.

Second, even if the plasma appears macroscopically steady, the distribution function itself may not be so. In fact, writing $\frac{\partial F}{\partial t} = 0$ implies that anomalous diffusion is unimportant, diffusion being produced only by collitions. Anomalous diffusion is due to turbulent behavior in a

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(*) One can excite oscillations in the plasma by oscillating a probe at high frequencies, however.

microscopic scale (for F) or a macroscopic scale (for the moments of F). As the Navier-Stokes equations of Fluid Mechanics are understood to contain the phenomena of turbulence and there is no need to revert to molecular theory to explain them, one expects that the B-L equation is still valid. But the difficulty of turbulence stems from a non-steady behavior around an average steady state in the presence of steady boundary conditions.

If one retains $\frac{\partial F}{\partial t}$, no information is available to regulate this fluctuating, non-evolutive time dependence. On the other hand, if one averages over some period of time large enough so that $\left(\frac{\partial F}{\partial t}\right)_{av} = 0$, the non-linear term with the macroscopic self-consistent field and the non-linear collision operator have an unknown form. The only possible way out is some consistent analysis of the spectrum of fluctuation (see [45]).

The validity of the assumption $\frac{\partial F}{\partial t} = 0$ has created much controversy in the past. Nevertheless, it can be said roughly that hot plasmas, certain far-from-equilibrium low-temperature plasmas, and gas discharges (where even macroscopic fluctuations are observed) obey anomalous diffusion, while decaying plasmas and quiescent, thermal, alkali plasmas seem to obey classical diffusion. See reviews by Hoh [46], Robertson and Pardo [47].

D. Resume:

Under the above conditions the following equations explicitly describe the problem to be considered:

$$\left\{ w_z \frac{\partial}{\partial z} + w_\xi \frac{\partial}{\partial \xi} + \frac{e}{m_e}\frac{\partial V}{\partial z}\frac{\partial}{\partial w_z} + \frac{e}{m_e}\frac{\partial V}{\partial \xi}\frac{\partial}{\partial w_\xi} - \frac{eB}{c\,m_e}\left(w_\eta \frac{\partial}{\partial w_\xi} - w_\xi \frac{\partial}{\partial w_\eta}\right) + s\left(\frac{w_\eta^2}{\xi}\frac{\partial}{\partial w_\xi} - \frac{w_\eta w_\xi}{\xi}\frac{\partial}{\partial w_\eta}\right)\right\} F^e = \frac{\delta F^e}{\delta t} \qquad (1.4)$$

$$\left\{ w_z \frac{\partial}{\partial z} + w_\xi \frac{\partial}{\partial \xi} - \frac{Z_i e}{m_i}\frac{\partial V}{\partial z}\frac{\partial}{\partial w_z} - \frac{Z_i e}{m_i}\frac{\partial V}{\partial \xi}\frac{\partial}{\partial w_\xi} + \frac{Z_i eB}{c\,m_i}\left(w_\eta \frac{\partial}{\partial w_\xi} - w_\xi \frac{\partial}{\partial w_\eta}\right) + \left(s\,\frac{w_\eta^2}{\xi}\frac{\partial}{\partial w_\xi} - \frac{w_\eta w_\xi}{\xi}\frac{\partial}{\partial w_\eta}\right)\right\} F^i = \frac{\delta F^i}{\delta t} \qquad (1.5)$$

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{1}{\xi^s}\frac{\partial}{\partial \xi}\,\xi^s \frac{\partial}{\partial \xi}\right\} V = 4\pi e (N_e - Z_i N_i) \qquad (1.6)$$

$$\left.\begin{aligned} N_e &= \int F^e d\vec{w} \\ N_i &= \int F^i d\vec{w} \end{aligned}\right\} \qquad (1.7)$$

The left hand side of (1.4) and (1.5) are derived in Appendix A. For $s = 1$ (axisymmetric problem) the coordinates are (z, ξ, ψ) (z, r, φ), (Fig. 3). For the strip, $(s = 0)$, we have (z, ξ, ψ) (z, x, y), (Fig. 4).

It will sometimes be convenient to use, instead of w_ξ, w_ψ, the quantities $w_\perp, \omega$, where

$$w_\xi = w_\perp \cos\omega \, , \quad w_\psi = w_\perp \sin\omega \qquad (1.8)$$

(The magnetic field is along z-axis, i.e., $w_\parallel \equiv w_z$).

Then one has the identity

$$\frac{\partial}{\partial\omega} \equiv w_\xi \frac{\partial}{\partial w_\psi} - w_\psi \frac{\partial}{\partial w_\xi} \qquad (1.9)$$

The boundary conditions for Eq. (1.4) - (1.7) are:

$$\left.\begin{aligned} &V = V_p \quad \text{on the probe} \\ &\lim_{\rho\equiv(z^2+\xi^2)^{1/2}\to\infty} V = 0 \end{aligned}\right\} \qquad (1.10)$$

Space potential is taken here as origin of V_p so that $V_p = 0$ corresponds to V_s of Fig. 1; and F^e and F^i are known quantities for $\rho \to \infty$. In particular

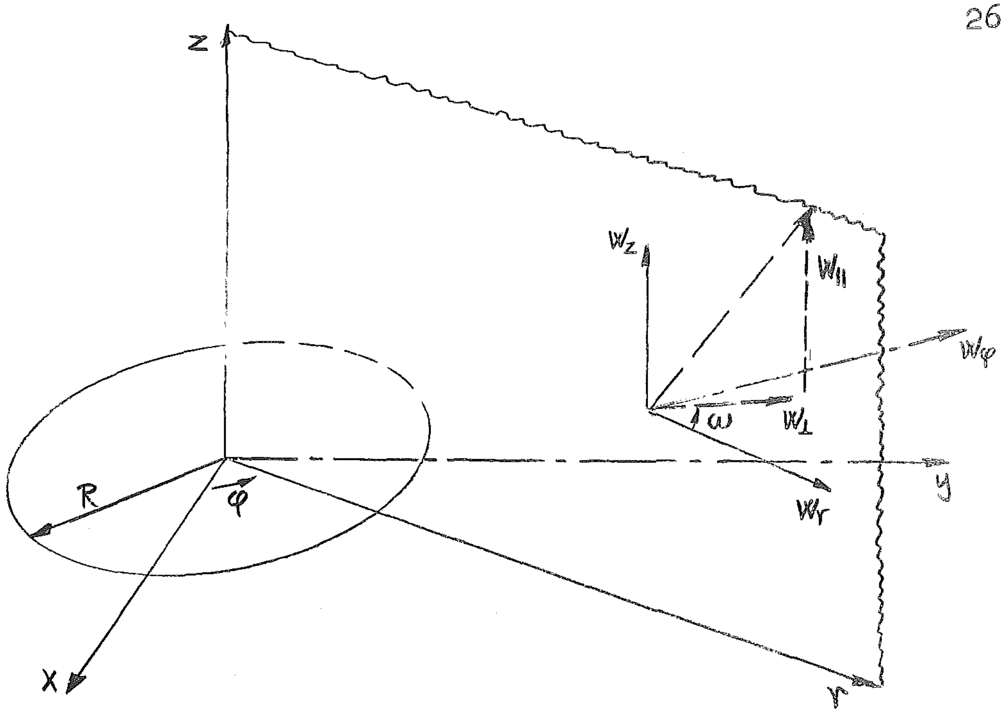

Fig. 3. Disc; nomenclature for physical and velocity space coordinates.

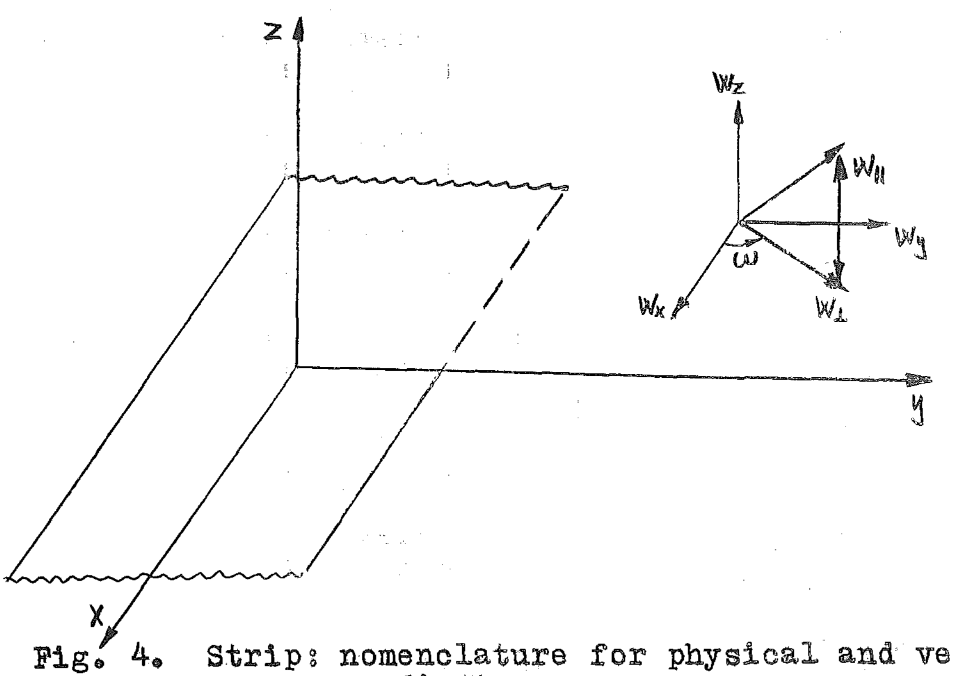

Fig. 4. Strip: nomenclature for physical and velocity space coordinates.

$$
\begin{aligned}
&\lim_{\rho \to \infty} N_e = N_\infty \\
&\lim_{\rho \to \infty} N_i = \frac{N_\infty}{Z_i} \\
&\lim_{\rho \to \infty} \int \frac{m_e}{2} w^2 F^e d\vec{w} = \frac{3}{2} N_\infty k T_e \\
&\lim_{\rho \to \infty} \int \frac{m_i}{2} w^2 F^i d\vec{w} = \frac{3}{2} \frac{N_\infty}{Z_i} k T_i
\end{aligned}
\tag{1.11}
$$

Although we shall study the distribution functions F^e, F^i, we are most interested in the currents to the probe $I_e(V_p), I_i(V_p)$, so that we do not even need the detailed spatial variation $V(r), N_e(r)$ $N_i(r)$. However, it is necessary to know, partially at least, the behavior of F^e, F^i to find I_e, I_i.

We shall consider the different regions of the characteristic in Ch. IV. An important by-product of this study will be the determination of the region disturbed by the probe and the dependence of the disturbed region on the different parameters that appear in the problem.

Chapter II

Mathematical Formulation

A. Non-dimensional equations

In order to obtain significant non-dimensional equations, we now define non-dimensional dependent variables which are of order of unity over some region of physical space. However, we have two different sets of independent variables: (z, ξ, η), (w_z, w_ξ, w_η). Because the analysis in (z, ξ, η) will prove to be rather complicated it is convenient to avoid considering the infinite range of $\vec{w}$ when making an analysis of orders of magnitude. We assume, therefore, that the high-energy tails of F^e, F^i do not produce any special effects (e.g., the way Landau damping or runaway electrons appear in other problems). This assumption is strongly justified on the basis that the electric field imposed by the probe is the cause of the perturbations existing and the field is governed by Poisson's equation which is sensitive only to the densities.

This allows us to restrict the consideration of characteristic magnitudes to the space coordinates.

(As we shall see, there are five important lengths in the problem). We define:

$$f^e = \frac{F^e U_e^3}{N_{\infty}} , \qquad f^i = \frac{F^i U_i^3}{N_{\infty}/Z_i} \tag{2.1}$$

$$\phi = \frac{V}{V_p} , \qquad \frac{\vec{W}}{U} = \vec{v} \tag{2.2}$$

where

$$U_j^2 m_j = kT_j$$

Combining the above definitions with (1.4) - (1.11) we have the following set of equations:

$$\left\{ v_z \frac{\partial}{\partial z} + v_\xi \frac{\partial}{\partial \xi} + \alpha \frac{\partial \phi}{\partial z} \frac{\partial}{\partial v_z} + \alpha \frac{\partial \phi}{\partial \xi} \frac{\partial}{\partial v_\xi} - \left(s \frac{v_\psi}{\xi} - \frac{1}{\ell_e} \right) \frac{\partial}{\partial w} \right\} f^e = \frac{1}{\lambda} \frac{\delta f^e}{\delta t} \tag{2.3}$$

$$\left\{ v_z \frac{\partial}{\partial z} + v_\xi \frac{\partial}{\partial \xi} - \frac{\alpha}{\beta} \frac{\partial \phi}{\partial z} \frac{\partial}{\partial v_z} - \frac{\alpha}{\beta} \frac{\partial \phi}{\partial \xi} \frac{\partial}{\partial v_\xi} - \left(s \frac{v_\psi}{\xi} + \frac{1}{\ell_i} \right) \frac{\partial}{\partial w} \right\} f^i = \frac{1}{\lambda} \frac{\delta f^i}{\delta t} \tag{2.4}$$

$$\alpha \lambda_D^2 \left\{ \frac{\partial^2}{\partial z^2} + \frac{1}{\xi^s} \frac{\partial}{\partial \xi} \xi^s \frac{\partial}{\partial \xi} \right\} \phi = n^e - n^i \tag{2.5}$$

where

$$\alpha = \frac{eV_p}{kT_e}\,, \qquad \beta = \frac{T_i}{Z_i T_e} \tag{2.6}$$

$$n^e = \int f^e d\vec{v} = \frac{N_e}{N_\infty}\,, \qquad n^i = \int f^i d\vec{v} = \frac{N_i}{N_\infty/Z_i} \tag{2.7}$$

and

$$\ell_e = \frac{m_e U_e c}{eB}\,, \qquad \ell_i = \frac{m_i U_i c}{Z_i eB} \tag{2.8}$$

The boundary conditions have the normalized form

$$\left.\begin{array}{ll} \phi = 1 & \text{on the probe} \\ \lim\limits_{\rho\to\infty} \phi = 0 & \end{array}\right\} \tag{2.9}$$

and f^e, f^i are given for $\rho \to \infty$; in particular

$$\left.\begin{array}{ll} \lim\limits_{\rho\to\infty} n^e = 1\,, & \lim\limits_{\rho\to\infty} n^i = 1 \\ \lim\limits_{\rho\to\infty} \int f^e \frac{v^2}{2} d\vec{v} = \frac{3}{2}\,, & \lim\limits_{\rho\to\infty} f^i \frac{v^2}{2} d\vec{v} = \frac{3}{2} \end{array}\right\} \tag{2.10}$$

Also, the identities

$$\left.\begin{array}{l} \frac{1}{\lambda}\frac{\delta f^e}{\delta t} \equiv \frac{U_e^2}{N_\infty}\frac{\delta F^e}{\delta t} \\ \frac{1}{\lambda}\frac{\delta f^i}{\delta t} \equiv \frac{U_i^2}{N_\infty/Z_i}\frac{\delta F^i}{\delta t} \end{array}\right\} \tag{2.11}$$

define the non-dimensional collision operators, which are of order of unity (for $|\vec{V}| = O(1)$).

The equations for the first two moments of f^e, f^i are given:

$$\frac{\partial}{\partial \vec{r}} \cdot n^e \vec{u}^e = 0, \quad \frac{\partial}{\partial \vec{r}} \cdot n^i \vec{u}^i = 0 \qquad (2.12)$$

$$\left.\begin{aligned} n^e \vec{u}^e \cdot \frac{\partial}{\partial \vec{r}} \vec{u}^e &= -\frac{\partial}{\partial \vec{r}} \cdot \overline{\overline{P}}^e + n^e \alpha \frac{\partial \phi}{\partial \vec{r}} - \frac{1}{\ell_e} n^e \vec{u}^e \times \vec{1}_z + \frac{\overline{\overline{T}}^e}{\lambda} \qquad (2.13) \\ n^i u^i \cdot \frac{\partial}{\partial \vec{r}} \vec{u}^i &= -\frac{\partial}{\partial \vec{r}} \cdot \overline{\overline{P}}^i - n^i \frac{\alpha}{\beta} \frac{\partial \phi}{\partial \vec{r}} + \frac{1}{\ell_i} n^i \vec{u}^i \times \vec{1}_z + \frac{\overline{\overline{T}}^i}{\lambda} \end{aligned}\right\}$$

where

$$n^j \vec{u}^j = \int f^j \, \vec{v} \, d\vec{v}, \quad \overline{\overline{P}}^j = \int f^j (\vec{v} - \vec{u}^j)^2 \, d\vec{v}$$

$$\overline{\overline{T}}^j = \int \frac{\delta f^j}{\delta t} \, \vec{v} \, d\vec{v} \qquad (2.14)$$

and $\vec{1}_z$ is the unit vector along the z-axis.

B. The characteristic lengths

Eqs. (2.3 - (2.5), (2.12) and (2.13) are now non-dimensional except for factors of dimension $(\text{length})^{-1}$. Five lengths appear explicitly in them: λ_D, λ, ℓ_e, ℓ_i and R. We can form four non-dimensional parameters with them

$$\left.\begin{aligned} \frac{\ell_e}{\ell_i} &= \left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{Z_i}{\beta}\right)^{1/2} \equiv \mu \\ \frac{\lambda_D}{\lambda} &= \frac{\Lambda}{4\pi/3\, b \ln\Lambda} \equiv \varepsilon \\ \frac{\ell_e}{R} &= \sigma \\ \frac{\lambda_D}{\ell_e} &= \gamma \end{aligned}\right\} \quad (2.15)$$

μ and ε are natural small parameters of any fully-ionized classical plasma. $\left(\frac{Z_i}{\beta}\right)^{1/2} = Z_i \left(\frac{T_e}{T_i}\right)^{1/2}$ will be assumed to be O(1).

σ serves to classify magnetic fields in the context of probe theory.

a) Weak B $\quad \sigma^{-1} \ll 1$

b) Moderate B $\quad \sigma^{-1} = O(1)$

c) Strong B $\quad \sigma^{-1} \gg 1 \quad \frac{\sigma}{\mu} \gg 1$

d) Very strong B $\quad \sigma^{-1} \gg 1 \quad \frac{\sigma}{\mu} \ll 1$

Case a) is of little interest because, to zero order in σ^{-1}, the I - V_p diagram should not change at all. (It appears empirically that no sensible change exists even for the less strict inequality $\sigma^{-1} < 1$. (This is discussed from a mathematical viewpoint in Ch. V).

Cases b) and c) are the most interesting. The present paper is concerned with case c); however, we shall find out in Ch. V that there is a possibility of the results being used for the preceding cases too.

Case d) is somewhat unrealistic for the values of B and R commonly encountered in the laboratory; some cases of astrophysical interest may exist however.

We require also the inequality

$$\frac{\ell_e}{\lambda} \equiv \gamma^{-1}\varepsilon \ll 1 \qquad (2.16)$$

to be satisfied; otherwise no sensible modification of the probe characteristic would exist. Also the commonly-found inequality will be assumed

$$\frac{\lambda_D}{R} \equiv \gamma\sigma \ll 1 \qquad (2.17)$$

The restriction $\frac{\sigma}{\mu} \equiv \frac{\ell_i}{R} \gg 1$ can normally be relaxed without degrading the quantitative accuracy of the solution; the extent to which this is true is difficult to determine analytically, but the present results should be essentially correct for all cases satisfying $\frac{\sigma}{\mu} > 1$ (see Ch. IV).

The ratios $\frac{\lambda_D}{\ell_e} = \gamma$ and $\frac{\lambda_D}{R} \equiv \varepsilon^{-1}\gamma\sigma$ are left arbitrary; there are thus two "small" lengths ℓ_e, λ_D and two "large" lengths λ and R. No relative ordering is imposed between them so that the influence of the different parameters will appear clearly in the solution.

Finally, the magnitude of α will depend upon the region of the characteristic considered.

Let us now estimate the magnetic field $\vec{B}'$ produced by the currents. From Maxwell's equation

$$\nabla \times \vec{B}' = -\frac{4\pi}{c}\vec{J}$$

and the induced field is of order

$$B' = O\left(\frac{4\pi}{c}\frac{\ell_e}{R} e N_\infty \left(\frac{kT_e}{m_e}\right)^{1/2} \frac{R\lambda}{\ell_e}\right)$$

(This relation anticipates the results of the next chapter where it is found that the electron current density is, at most, of order $\frac{\ell_e}{R} N_\infty \left(\frac{kT_e}{m_e}\right)^{1/2}$ and the maximum length of relevance of order $\lambda \frac{R}{\ell_e}$). Therefore

$$\frac{B'}{B} = O\left(\frac{kT_e}{m_e c^2}\frac{\ell_e \lambda}{\lambda_D^2}\right)$$

Of course the first factor is here enormously small. At $T_e \approx 2 \times 10^3$, $\frac{kT_e}{m_e c^2} \approx 3.3 \times 10^{-7}$. For $T_e < 10^5$ K, $\frac{kT_e}{m_e c^2} < 1.7 \times 10^{-5}$. For temperatures less than this limit, it will be seen in Appendix C that the ratio $\frac{B'}{B}$ is indeed very small.

Figs. 5 and 6 illustrate the domains of probe solutions. In Fig. 5 are given the regions of validity

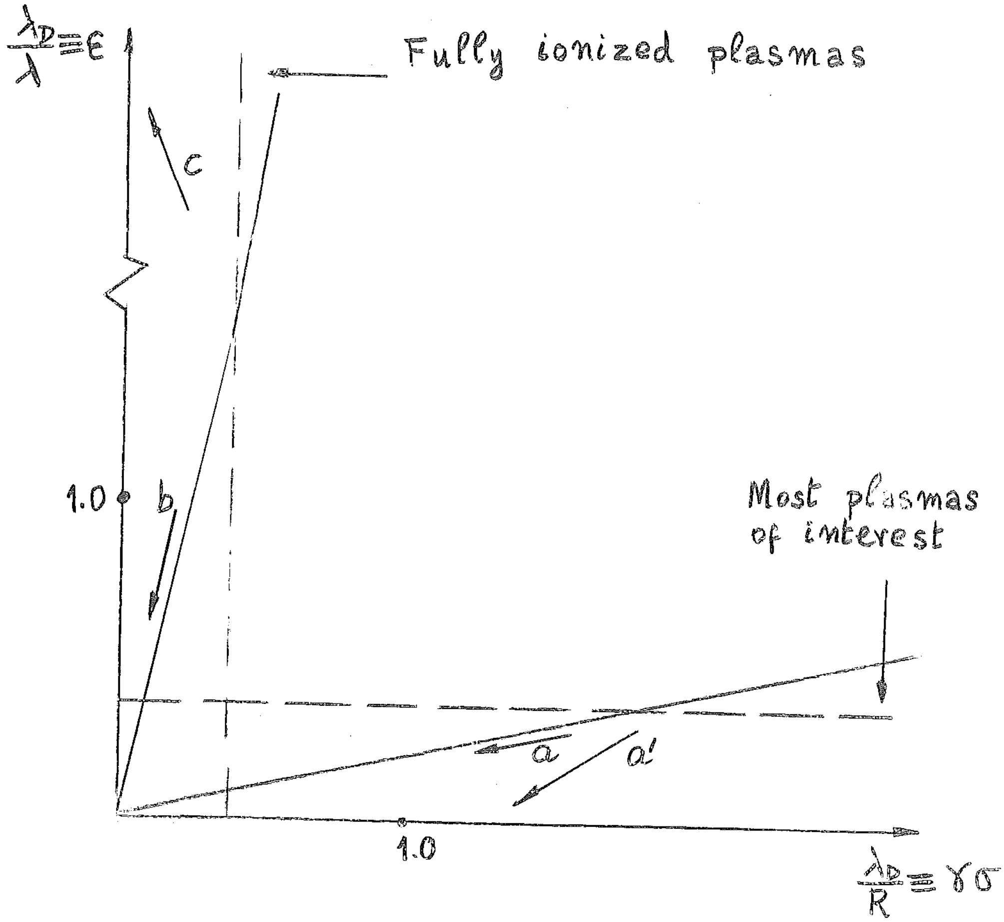

Fig. 5. Probe regimes for $\sigma^{-1} = 0$ $(B \approx 0)$.
Dilute plasmas: a ($\lambda_D \ll R \ll \lambda$) and a' ($\lambda_D \approx R \ll \lambda$).
Dense plasmas: b ($\lambda_D \ll \lambda \ll R$).
Continuum plasmas: c ($\lambda \ll \lambda \ll R$).

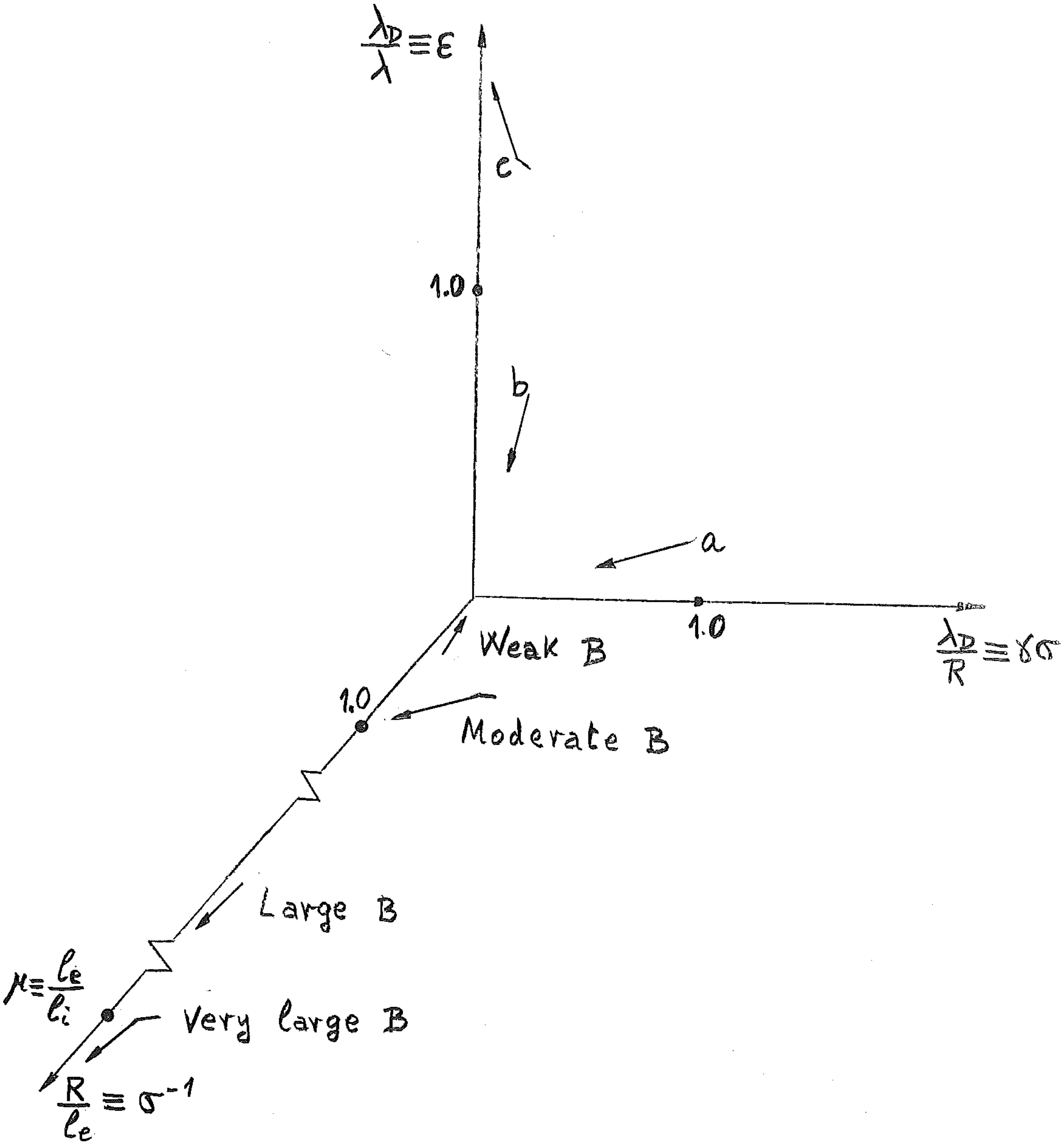

Fig. 6. Probe regimes for $\sigma^{-1} > 0$ (B > 0).

of previous theories for $B \approx 0$(*); i.e., for dilute, dense and continuum plasmas. In Fig. 6 the representation of Fig. 6 is extended to the case $B \neq 0$; the plane $\sigma^{-1} = 0$ reproduces Fig. 5.

Appendix C is included to give typical ranges of the four non-dimensional parameters for the low-temperature, fully ionized C_s plasma produced by a Q-machine.

C. Perturbation method of solution

Consider the equation

$$P(\vec{x}, g, \varepsilon) = 0$$

where P is a differential operator and ε is a small parameter. Let $g(\vec{x}, \varepsilon)$ be its solution

$$P(\vec{x}, g(\vec{x}, \varepsilon), \varepsilon) \equiv 0$$

Assuming a Taylor expansion to exist for the dependence of g on ε, we would write

---

(*) A similar diagram was given by Su and Probstein 2.

$$g = g(\vec{x},\varepsilon)\Big|_{\varepsilon=0} + \varepsilon \frac{\partial g}{\partial \varepsilon}\Big|_{\varepsilon=0} + \cdots \qquad (2.18)$$

which is normally written as

$$g = g_0(x) + \varepsilon g_1(x) + \cdots \qquad (2.19)$$

The g_i's are expected to have a regular behavior in the domain of $\vec{x}$; then for ε small enough, the n^{th} term is negligible as compared to the $(n - 1)^{th}$ one. It is hoped, furthermore, that if all small or large parameters have been taken into account explicitly, one can retain ε small but finite without impairing the relative magnitude of successive terms.

When such an expansion fails because g_j becomes infinite in some region of $\vec{x}$-space, one speaks of a singular, as opposed to regular, perturbation problem. A large body of literature exists on this subject. We shall consider here one specific method, that of multiple scales, which seems most appropriate because of the many characteristic lengths of the present problem. In the 1930's, Krylov and Bogoliubov [48] developed and applied successfully to Non-Linear Mechanics, Lagrange's "slow coefficients" method of Celestial Mechanics. It was "rediscovered" in 1962 [49] and independently applied to non-equilibrium statistical mechanics by Frieman [50] and Sandri [51]. They introduce a set of progressively slower time scales. The forms of the kinetic equations obtained in the "slow variables" result explicitly from the requirement that secularity must be avoided in the equations for the "fast variables".

Insight into the method can be gained by considering the following example:

One is looking for a splitting of the dependence of the solution on time (t) so as to avoid the appearance of secular terms. This means that, for example, the function

$$g(t) \equiv \cos t \, e^{-\varepsilon t}$$

can be thought to have the form

$$g(t) \equiv \varphi(t, \varepsilon t)$$

where $\varphi(\circ, \circ)$ is, of course, given by

$$\varphi(x, y) = \cos x \, e^{-y}$$

Such functional separation is unknown beforehand and must be found in the course of the solution by eliminating secularities. Now in the present example (and it is a fairly general feature of singular perturbation problems with a physical origin), two characteristic times appear explicitly in the equations: t_0 and t_1.

It might be guessed that the splitting looked for is simply

$$g(t) = g\left(\frac{t}{t_0}, \frac{t}{t_1}\right) = g\left(\frac{t}{t_0}, \varepsilon\frac{t}{t_0}\right) =$$

$$= g\left(\tau_0, \tau_1\right) = g\left(\tau_0, \varepsilon\tau_0\right)$$

where the quantities

$$\tau_0 = \frac{t}{t_0}, \quad \tau_1 = \frac{t}{t_1}, \quad \varepsilon = \frac{t_0}{t_1}$$

are non-dimensional.

If one uses the simple procedure of Eqs. (2.18) and (2.19) the result is

$$g(t) = g(\tau_0, 0) + \varepsilon\frac{t}{t_0}\left.\frac{\partial g}{\partial \tau_1}\right|_{\varepsilon=0} + \ldots =$$

$$= g_0 + \varepsilon g_1 + \ldots$$

This expansion becomes useless for $t \gg t_0$ because of the appearance of secular terms in g_j, $j \geqslant 1$; i.e., g_j is not much smaller than g_{j-1} for $t \approx \frac{t_0}{\varepsilon} = t_1$.

Of course a more complete form of the function g should be written as

$$g = g\left(\frac{t}{t_0}, \frac{t}{t_1}, \frac{t_0}{t_1}\right) = g\left(\tau_0, \varepsilon\tau_0, \varepsilon\right)$$

and the usual expansion can be used for the functional dependence on ε which does not involve t. The dependence on $\tau_0\varepsilon$ is handled by expanding all the derivatives as follows

$$\frac{\partial}{\partial t} = \frac{\partial\tau_0}{\partial t}\frac{\partial}{\partial\tau_0} + \frac{\partial\tau_1}{\partial t}\frac{\partial}{\partial\tau_1} + \cdots$$
$$= \frac{1}{t_0}\frac{\partial}{\partial\tau_0} + \frac{1}{t_1}\frac{\partial}{\partial\tau_1} + \cdots$$
$$= \frac{1}{t_0}\left(\frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1} + \cdots\right)$$

The quantities within the brackets are now non-dimensional; otherwise, the dependence on $\tau_1 \equiv \varepsilon\tau_0$ is retained completely.

We should point out: first, in singular perturbation problems fractional powers and logarithms of the small parameter often appear; and second, if several small parameters are present, as is the case in this thesis, the sequence of terms in the expansion is not evident a priori. We should use the more general form (going back to the above example, with several ε's).

$$\tau_j = \frac{t}{t_0}\,\delta_j(\varepsilon_1, \varepsilon_2, \ldots) \qquad (2.20)$$

where ε_1, ε_2, ... are the small parameters and the δ_j's constitute an unknown sequence of function of them, ordered in an asymptotic way:

$$\lim_{(\varepsilon_1, \varepsilon_2, \ldots) \to \infty} \frac{\delta_j}{\delta_{j-1}} = 0$$

The above limits hold hopefully independently of the order in which $\varepsilon_1, \varepsilon_2 \ldots$ are allowed to go to zero. To illustrate the expansion procedure and to show that space-dependent problems as well as time-dependent problems are amenable to this technique, the method is applied in Appendix D to the determination of thermal equilibrium correlations in a plasma.

Finally, we call attention to a significant variation of the standard multiple-scales technique which is necessary in this problem. Normally when one solves the set of equations obtained through an asymptotic expansion, the solutions are sequential in the sense that the solution of the n^{th} order equation requires only information generated by the solutions of the lower order equations. In this problem, a difficult coupling of the equations exists such that, when obtaining a solution in one physical scale, it is necessary to have information from the following higher-order scales. (The physical origin of this difficulty is immediately apparent: boundary conditions are given in different domains of the independent-variable space.)

In this connection the following idea was used. If we have an equation like

$$\frac{\partial g}{\partial \tau_j} = A\left[g(\tau_0,\ldots \tau_j,\ldots),\tau_0,\ldots\tau_j\ldots\right] \quad (2.21)$$

integrating over τ_j we have

$$g = g(\tau_j = 0) + \int_0^{\tau_j} A(\tau_j)\,d\tau_j$$

If we require a) g to be finite as $\tau_j \to \infty$ and b) $\frac{\partial g}{\partial \tau_j}$ not to change signs an infinite number of times, then both of the following equalities must hold:

$$\lim_{\tau_j\to\infty} \frac{\partial g}{\partial \tau_j} = 0, \qquad \lim_{\tau_j\to\infty} A = 0 \qquad \text{(see (2.21))}$$

Therefore, we systematically will write from equations like (2.21):

$$A\left[g(\infty,\ldots\infty,\tau_j,\ldots),\infty,\ldots\infty,\tau_j,\ldots\right] = 0$$

Requirement a) is not obviously imposed by the problem and will have to be discussed later. Requirement b) is unsatisfactory when τ is a time variable since fluctuations and instabilities do occur in such problems.

It is appropriate if τ is a space variable and a minimum is known about the character of the solution.

This idea does not provide enough information in many problems (see Appendix D); however, whenever usable, is much more simple than the common technique.

Chapter III

Character of the Solution

A. Introduction

The basic result that the present study seeks to obtain is the total current to the probe, I, as a function of the probe potential, V_p. This is the I - V_p diagram and will be considered in detail in the following chapter.

The present one is concerned with and will provide two important results. The first is to derive a partial differential equation which governs the electron current collected. The solution of this equation is described in Chapter IV; over most of the diagram, numerical computations have to be used. Because of the large ratio of ion to electron mass (μ very small), the ion current, I^i, is negligible for a broad range of α. Where this is not so (for V_p approximately equal to or less than the floating potential, V_f), I^i will be shown not to differ from its value for $B = 0$, for which case the theories mentioned in Section 1-B are available and will be used. Therefore, by obtaining the electron current, I^e, it will be possible to describe the whole probe characteristic.

The second result is a description of the perturbation produced by the probe. In order to obtain I^e, a detailed analysis of the entire space disturbed by the probe is required. This space will prove to consist of several regions or layers where the equations exhibit different behavior. There appear phenomena which are caused by the presence of the magnetic field and which are themselves of great interest.

Before proceeding to the mathematical analysis, we give in the next section a brief description of the character of the solution as deduced from simple considerations. In the last section of this chapter, a more detailed description is presented, in the light of the intermediate results.

B. The channeling effect

Consider a spherical probe at space potential in the absence of a magnetic field. The thermal flux to the probe surface decays as ρ^{-2}, where ρ is the distance to the center of the probe. This simple relation stems from a first, obvious integral of the continuity equation for any of the species. If the probe is small enough, its influence spreads sufficiently rapidly to be negligible at large distances.

However, when B is large, the electrons at least are inhibited from flowing across field lines. The flux along $\vec{B}$ remains nearly constant over distances of many mean free paths and transport coefficients will come into play. Moreover, in the plane of the probe ($z = 0$) and outside its area, the flux along $\vec{B}$ is zero by symmetry but it is not so inside. Therefore, strong gradients will appear around $\xi = R$, for both the disc and the strip. The way in which they are smoothed out is also of interest. This region around $\xi = R$, the boundary of the shadow(*), will be considered in Appendix F. In this chapter, it will be studied only very far from the probe where the gradients are not strong any longer; the more detailed analysis of Appendix F is not necessary for the leading behavior of the solution.

Finally, Poisson's equation is expected to be unimportant. In effect, quasineutrality should exist very far from the probe. Near the probe the problem is almost unidimensional so that the detailed form of the potential is irrelevant.

(*) We shall call the interior of an imaginary cylindrical surface whose cross section is that of the probe and whose generatrices are parallel to $\vec{B}$, the "shadow" of the probe.

C. The expanded equations

Throughout this chapter a finite range of V_p (measured relative to V_s as mentioned in Chapter II) will be considered. By this is meant that the normalized probe potential, $\alpha \equiv \frac{eV_p}{kT_e}$, will not enter the asymptotic process. Extreme ranges of α will be considered in Chapter IV.

The equations to be solved are those for f^e, f^i and ϕ given in Section II-A. Upon multiplying by ℓ_e and ordering the terms in a convenient way we obtain for f^e and f^i

$$\frac{\partial f^e}{\partial \omega} + \ell_e \left\{ V_z \frac{\partial}{\partial z} + \alpha \frac{\partial \phi}{\partial z} \frac{\partial}{\partial V_z} \right\} f^e + \ell_e \left\{ V_\xi \frac{\partial}{\partial \xi} + \alpha \frac{\partial \phi}{\partial \xi} \frac{\partial}{\partial V_\xi} \right\} f^e - s\,\ell_e \frac{V_\eta}{\xi} \frac{\partial f^e}{\partial \omega} = \gamma^{-1} \varepsilon \frac{\delta f^e}{\delta t} \tag{3.1}$$

$$\ell_e \left\{ V_z \frac{\partial}{\partial z} - \frac{\alpha}{\beta} \frac{\partial \phi}{\partial z} \frac{\partial}{\partial V_z} \right\} f^i + \ell_e \left\{ V_\xi \frac{\partial}{\partial \xi} - \frac{\alpha}{\beta} \frac{\partial \phi}{\partial \xi} \frac{\partial}{\partial V_\xi} \right\} f^i - s\,\ell_e \frac{V_\eta}{\xi} \frac{\partial f^i}{\partial \omega} - \mu \frac{\partial f^i}{\partial \omega} = \gamma^{-1} \varepsilon \frac{\delta f^i}{\delta t} \tag{3.2}$$

and for ϕ

$$\alpha \lambda_D^2 \left\{ \frac{\partial^2}{\partial z^2} + \frac{1}{\xi^s} \frac{\partial}{\partial \xi} \xi^s \frac{\partial}{\partial \xi} \right\} \phi = n^e - n^i \tag{3.3}$$

The bracket in (3.3) is the Laplace operator and $s = 0,1$ for cartesian and cylindrical coordinates respectively. The angle ω in velocity space has been defined in Section I-D and the parameters $\mu, \varepsilon, \sigma, \gamma$ in Section II-B

Now, as in the example considered in Section II-C (see Eq. (2.20), we define the sets of non-dimensional space variables, ξ_j and z_j :

$$\xi_j \equiv \frac{\xi - R}{\ell_e} \delta_j^{\xi} \ , (j < K); \quad \xi_j \equiv \frac{\xi}{R} \delta_j^{\xi}, \quad (j \geq K)$$

$$z_j \equiv \frac{z}{\ell_e} \delta_j^{z} \qquad (3.4)$$

The index K will hereafter be used to denote the first non-dimensional length-scale along the ξ-axis which has a characteristic length R or greater. For $j < K$ we have introduced R in the definition of ξ_j to allow for the presence of strong gradients around $\xi = R$ (*); for $j \geqslant K$ this is neither necessary nor convenient.

Every scale ξ_j, z_j has a characteristic length defined by

$$L_j = \frac{\ell_e}{\delta_j^{\xi}} \quad ; \qquad L_j^z = \frac{\ell_e}{\delta_j^z} \qquad (3.5)$$

(*) For $s = 0$, the region around $\xi = -R$ should be treated likewise; however, the symmetry around $\xi = 0$ allows to consider only the positive values of ξ.

with δ_j^{ξ}, δ_j^{z} being functions of $(\mu, \varepsilon, \sigma, \gamma)$ to be determined in the course of the solution and satisfying

$$\lim_{(\mu,\varepsilon,\sigma)\to 0} \frac{\delta_j^{\xi}}{\delta_{j-1}^{\xi}} = 0; \quad \lim_{(\mu,\varepsilon,\sigma)\to 0} \frac{\delta_j^{z}}{\delta_{j-1}^{z}} = 0 \tag{3.6}$$

Thus K is the value of j for which

$$L_K^{\xi} = R; \qquad \delta_K^{\xi} = \frac{\ell_e}{R} \equiv \sigma$$

We have chosen ℓ_e as our primary length; in general the smallest characteristic length is chosen. Then the δ's, besides satisfying (3.6), would satisfy

$$\lim_{(\varepsilon_1, \varepsilon_2, \ldots) \to 0} \delta_j = 0 \tag{3.7}$$

$(\varepsilon_1, \varepsilon_2, \ldots)$ being the small parameters of the problem considered. However because γ is left arbitrary ($\ell_e \ll \lambda_D$ or $\ell_e \gg \lambda_D$), our δ's in (3.5) do not necessarily satisfy an equation like (3.7) for all j.

Now from (3.4) we have

$$\frac{\partial}{\partial \xi} = \sum_j \frac{1}{L_j^{\xi}} \frac{\partial}{\partial \xi_j}; \qquad \frac{\partial}{\partial z} = \sum_j \frac{1}{L_j^{z}} \frac{\partial}{\partial z_j} \tag{3.8}$$

We also expand the functions

$$
\begin{aligned}
f^e &= f_0^e + \Delta_1^e f_1^e + \Delta_2^e f_2^e + \cdots \\
f^i &= f_0^i + \Delta_1^i f_1^i + \cdots \qquad\qquad (3.9) \\
\phi &= \phi_0 + \Delta_1^\phi \phi_1 + \cdots
\end{aligned}
$$

where the sets of functions of $(\mu, \varepsilon, \sigma, \gamma)$, $\Delta_j^e, \Delta_j^i, \Delta_j^\phi$ satisfy conditions analogous to those of (3.6) and (3.7). We have written $\Delta_0^e = \Delta_0^i = \Delta_0^\phi \equiv 1$ in (3.9) which implies that f^e, f^i, ϕ are of order unity. While this is the case somewhere in space (ϕ at the probe, f^e and f^i far from it) it is not obviously so everywhere. In fact we shall see in Section III-F that this is not always true; nevertheless, the corrections are very easily made and we shall use this assumption to begin with.

Then, using (3.8) and (3.9), the Eq's. (3.1) - (3.3) become

$$
\begin{aligned}
&\frac{\partial}{\partial\omega}\sum_p \Delta_p^e f_p^e + \sum_j \delta_j^z \left\{ v_z \frac{\partial}{\partial z_j} + \alpha \frac{\partial}{\partial z_j}\Big(\sum_p \Delta_p^\phi \phi_p\Big)\frac{\partial}{\partial v_z} \right\} \\
&\quad \times \sum \Delta_p^e f_p^e + \sum_j \delta_j^\xi \left\{ v_\xi \frac{\partial}{\partial \xi_j} + \alpha \frac{\partial}{\partial \xi_j}\Big(\sum_p \Delta_p^\phi \phi_p\Big)\frac{\partial}{\partial v_\xi} \right\} \\
&\quad \times \sum \Delta_p^e f_p^e - s\frac{l_e}{\xi} v_\eta \frac{\partial}{\partial\omega}\sum_p \Delta_p^e f_p^e = \gamma^{-1}\varepsilon\,\frac{\delta f^e}{\delta t} \qquad (3.10)
\end{aligned}
$$

$$\delta_j^z\left\{V_z\frac{\partial}{\partial z}-\frac{\alpha}{\beta}\frac{\partial}{\partial z}\sum\Delta_p\phi_p\frac{\partial}{\partial V_z}\right\}\sum\Delta_p^i f_p^i+\sum_j\delta_j^\xi$$

$$\times\left\{V_\xi\frac{\partial}{\partial\xi_j}-\frac{\alpha}{\beta}\frac{\partial}{\partial\xi_j}\left(\sum_p\Delta_p^\phi\phi_p\right)\frac{\partial}{\partial V_\xi}\sum_p\Delta_p^i f_p^i\right\}$$

$$-s\frac{l_e}{\xi}V_\gamma\frac{\partial}{\partial\omega}\sum_p\Delta_p^i f_p^i-\mu\frac{\partial}{\partial\omega}\sum_p\Delta_p^i f_p^i=\gamma^{-1}\varepsilon\frac{\delta f^i}{\delta t}$$

$$(3.11)$$

$$\alpha\gamma^2\left\{\left(\sum_j\delta_j^z\frac{\partial}{\partial z_j}\right)^2+\frac{1}{\xi^s}\sum_j\delta_j^\xi\frac{\partial}{\partial\xi_j}\xi^s\sum_{j'}\delta_{j'}^\xi\frac{\partial}{\partial\xi_{j'}}\right\}$$

$$\times\sum_p\Delta_p^\phi\phi_p=\sum_p\Delta_p^e n_p^e-\sum_p\Delta_p^i n_p^i\quad(3.12)$$

Whenever a coordinate appears explicitly, e.g. in the right-hand sides of Equations (3.10) - (3.12), we can use any of the non-dimensional variables in the corresponding set in (3.4). For instance we can substitute for

$$\xi=\frac{l_e}{\delta_j^\xi}\xi_j,\quad(j\geq K);\quad\xi=\frac{l_e}{\delta_j^\xi}\left(\xi_j+\frac{\delta_j^\xi l_e}{R}\right),\quad(j<k)\quad(3.13)$$

for any j. The actual choice will depend on which region of space we are considering.

From (3.10) - (3.12), equations for the expanded moments of f^e and f^i can be obtained very easily. For instance, the continuity equation has the form:

$$\sum_j \delta_j^z \frac{\partial}{\partial z_j} \sum_p \Delta_p^e n_p^e u_z^{ep}$$
$$\frac{1}{\xi^s} \sum_j \delta_j^\xi \frac{\partial}{\partial \xi_j} \xi^s \sum_p \Delta_p n_p^e u_\xi^{ep} \doteq 0 \quad (3.14)$$

Observe that in (3.14) no cross-products such as $n_{p_1}^e u^{ep_2}$ ($p_1 \neq p_2$) exist and the same is true in the other Maxwell transfer equations. In the equivalent continuity or Navier-Stokes equations of Fluid Mechanics such cross-products appear whenever an expansion is made; this is because these equations are non-linear in the magnitudes expanded. In our problem the distribution functions f^e, f^i are the quantities expanded and these cross-products come from terms in the kinetic equations which are linear in f^e, f^i. For instance, to expand $n^e\vec{u}^e$, we write:

$$n^e\vec{u}^e = \int f^e \,\vec{v}d\vec{v} = \int f_0^e \,\vec{v}d\vec{v} + \Delta_1^e \int f_1^e \,\vec{v}d\vec{v} + \cdots$$
$$= n_0^e \,\vec{u}^{e0} + \Delta_1^e \, n_1^e \,\vec{u}^{e1} + \cdots$$

D. The description in the main ξ-region

In this section we shall consider Eq. (3.10) - (3.12) everywhere excluding a thin region on both sides of the "shadow" boundary, (see Fig. 7). The width of this region will shrink to zero as $(\varepsilon, \sigma, \mu) \to 0$. Therefore if we know the current density to the probe inside the "shadow",

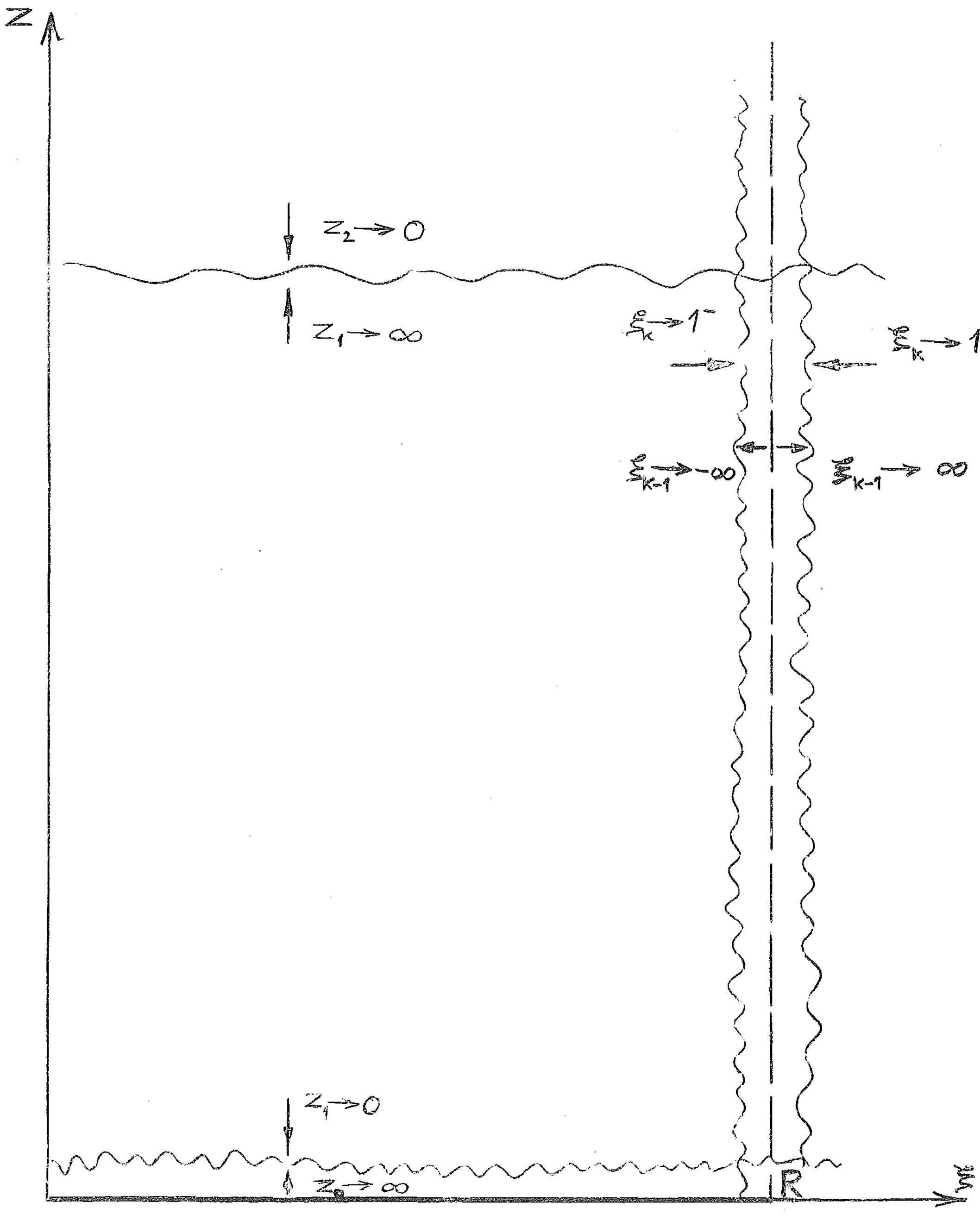

Fig. 7. The structure of the space around the probe.

integrate over the corresponding probe area, and let $(\varepsilon,\sigma,\mu)\to 0$ we shall obtain the current to the probe. When (ε,σ,μ) are small but finite, the result is the leading term of an asymptotic expansion in (ε,σ,μ).

Mathematically what we mean is that the limit $\xi_{K-1}\to\infty$ will be taken in Eqs. (3.10) - (3.12). This implies also $\xi_j\to\infty$ ($j < K - 1$) and, from the end of Section C in Chapter II, then:

$$\frac{\partial}{\partial\xi_j} = 0 \qquad (j < K)$$

We should point out, now, what is meant by the limit $\xi_{K-1}\to\infty$, with ξ_K finite. For this, consider Eq. (3.4); from it we can write:

$$\frac{\xi_{K-1}}{\xi_K} = \left(1 - \frac{R}{\xi}\right)\frac{\delta_{K-1}}{\delta_K} \qquad (3.15)$$

For any <u>fixed</u> ξ different than R, the left-hand side goes to infinity as $(\varepsilon,\sigma,\mu)\to 0$, as seen using Eq. (3.6). Therefore the set of points with finite ξ_K which have a non-infinite ξ_{K-1} (or ξ_j, $j < K - 1$) coordinate are reduced, in that limit, to the line $\xi = R$.

That "main" region which covers "most" (asymptotically) of the probe is described in the ξ_K scale;

we shall see in the following subsections that this ξ_K region splits into three z-layers.

The ξ_j $(j < K)$ regions are of no relevance for the leading behavior of the solution, except far from the probe, in the z_2-layer. In Appendix F these ξ interior regions are considered near the probe.

D1. The z_0-layer

Taking the limit $\xi_{K-1} \to \infty$ results in dropping all the derivatives $\frac{\partial}{\partial \xi_j}$ $(j < K)$, in (3.10) - (3.12). Therefore the only change in the equations consists in writing $\sum_{j \geq K} \delta_j^{\xi} \frac{\partial}{\partial \xi_j}$ for $\sum_j \delta_j^{\xi} \frac{\partial}{\partial \xi_j}$. Retaining the dominant terms in every summation of (3.10) - (3.12) gives then:

$$\frac{\partial f_0^e}{\partial \omega} + \delta_0^z \left\{ v_z \frac{\partial}{\partial z_0} + \alpha \frac{\partial \phi_0}{\partial z_0} \frac{\partial}{\partial v_z} \right\} f_0^e + \delta_K^{\xi} \left\{ v_\xi \frac{\partial}{\partial \xi_K} + \alpha \frac{\partial \phi_0}{\partial z_0} \frac{\partial}{\partial v_z} \right\} f_0^e - s\,\delta^{\xi} \frac{v_\xi}{\xi_K} \frac{\partial f_0^e}{\partial \omega} = \gamma^{-1} \varepsilon \left(\frac{\delta f^e}{\delta t} \right)_0 \tag{3.16}$$

$$\delta_0^z \left\{ v_z \frac{\partial}{\partial z_0} - \frac{\alpha}{\beta} \frac{\partial \phi_0}{\partial z_0} \frac{\partial}{\partial v_z} \right\} f_0^i + \delta_K^{\xi} \left\{ v_\xi \frac{\partial}{\partial \xi_K} - \frac{\alpha}{\beta} \frac{\partial \phi_0}{\partial \xi_K} \frac{\partial}{\partial v_\xi} \right\} f_0^i - s\,\delta_K^{\xi} \frac{v_\xi}{\xi_K} \frac{\partial f_0^i}{\partial \omega} - \mu \frac{\partial f_0^i}{\partial \omega} = \gamma^{-1} \varepsilon \left(\frac{\delta f^i}{\delta t} \right)_0 \tag{3.17}$$

$$\alpha \gamma^2 \left\{ \left(\delta_0^z\right)^2 \frac{\partial^2}{\partial z_0^2} + \frac{\left(\delta_K^{\xi}\right)^2}{\xi_K^s} \frac{\partial}{\partial \xi_K} \xi_K^s \frac{\partial}{\partial \xi_K} \right\} \phi_0 = n_0^e - n_0^i \tag{3.18}$$

where we have written $\xi = \frac{\ell_e}{\delta_k^{\xi}} \xi_k$, from (3.4).

We first want to show that L_0^z is of order λ_D independently of the value of γ. Observe that $\delta_k^{\xi} = \sigma$; for $\sigma \to 0$, (3.18) becomes

$$\alpha \left(\gamma \delta_0^z\right)^2 \frac{\partial^2 \phi_0}{\partial z_0^2} = n_0^e - n_0^i \qquad (3.19)$$

Suppose $L_0^z \ll \lambda_D$ (which is possible only if λ_D is <u>not</u> the smallest length of the problem, i.e., $\ell_e \ll \lambda_D$). Then

$$\gamma \delta_0^z \equiv \frac{\lambda_D}{L_0^z} \gg 1$$

and this would require in (3.19)

$$\frac{\partial^2 \phi_0}{\partial z_0^2} = 0 \qquad \text{or} \qquad \phi_0 = a z_0 + b$$

However our general condition

$$\lim_{z_0 \to \infty} \frac{\partial}{\partial z_0} \equiv 0$$

implies $a = 0$, and therefore ϕ_0 would be independent of z_0.

Similarly in (3.17), since $(\sigma, \mu, \gamma^{-1}\varepsilon) \to 0$, we get

$$v_z \frac{\partial f_0^i}{\partial z_0} - \frac{\alpha}{\beta} \frac{\partial \phi_0}{\partial z_0} \frac{\partial f_0^i}{\partial v_z} = 0$$

and from $\frac{\partial \phi_0}{\partial z_0} \equiv 0$, there results $\frac{\partial f_0^i}{\partial z_0} \equiv 0$.

Finally, from (3.16) we would get

$$\frac{\partial f_0^e}{\partial \omega} + \delta_0^z \left\{ v_z \frac{\partial}{\partial z_0} + \alpha \frac{\partial \phi_0}{\partial z_0} \frac{\partial}{\partial v_z} \right\} f_0^e = 0 \qquad (3.20)$$

or

$$\frac{\partial f_0^e}{\partial \omega} + \frac{\ell_e}{L_0^z} v_z \frac{\partial f_0^e}{\partial z_0} = 0 \qquad (3.21)$$

Thus $L_0^z \approx \ell_e$ and f_0^e is easily obtained:

$$f_0^e = f_0^e \left(v_z \omega - \frac{L_0^z}{\ell_e} z_0 \right)$$

But this function cannot be periodic in ω for a given z_0 if $\frac{\partial f_0^e}{\partial z_0} \rightarrow 0$ as $z_0 \rightarrow \infty$. Another way to look at this is to observe that, because

$$\lim_{z_0 \rightarrow \infty} \frac{\partial f_0^e}{\partial z_0} = 0$$

we must have $\frac{\partial f_0^e}{\partial \omega} = 0$ for very large z_0. Therefore the electrons must enter the z_0-region with a distribution function isotropic around the z-axis, and this isotropy is not modified either by the spiraling motion described by (3.21) or by the conditions on the probe (even if some reflection were to exist).

Therefore, no change exists on a ℓ_e-scale if $\ell_e \ll \lambda_D$. We write $\lambda_D = L_0^z$ (*) and the equations for this z_0-layer are

$$\frac{\partial f_0^e}{\partial \omega} = 0\, ; \qquad v_z \frac{\partial f_0^e}{\partial z_0} + \alpha \frac{\partial \phi_0}{\partial z_0} \frac{\partial f_0^e}{\partial v_z} = 0 \quad (3.22)$$

$$v_z \frac{\partial f_0^i}{\partial z_0} - \frac{\alpha}{\beta} \frac{\partial \phi_0}{\partial z_0} \frac{\partial f_0^i}{\partial v_z} = 0 \qquad (3.23)$$

$$\alpha \frac{\partial^2 \phi_0}{\partial z_0^2} = n_0^e - n_0^i \qquad (3.24)$$

The argument leading to $\frac{\partial f_0^e}{\partial \omega} = 0$ in (3.20) remains valid for $L_0^z = \lambda_D$.

Some simple results for the moments of f_0^e, f_0^i can be found from (3.22) and (3.23):

$$\frac{\partial}{\partial z_0} n_0^e u_z^{eo} = 0; \qquad \frac{\partial}{\partial z_0} n_0^i u_z^{io} = 0$$
$$u_\xi^{eo} = u_\eta^{eo} = P_{\xi\eta}^{eo} = P_{\xi z}^{eo} = P_{\eta z}^{eo} = 0 \qquad (3.25)$$

Eqs (3.22) - (3.24) will not be solved until Section F.

The equations for the higher-order terms f_P^e, f_P^i, ϕ_P $(P \geq 1)$ in the z_0-scale can be found by taking

---

(*) Numerical factors are excluded from the definitions of L_j, L_j^z and the δ's and Δ's for simplicity, since they are arbitrary.

terms of successive orders of magnitude in (3.10) - (3.12).

D2. The z_1-layer

Being interested in the solution to the lowest order in $(\varepsilon, \sigma, \mu)$, we proceed to formulate the equations for f_0^e, f_0^i, Φ_0 in the following layers. This amounts to taking the terms of next order in (3.10) - (3.12) and letting $z_0 \rightarrow \infty$.

For Poisson's equation, (3.24) provides still information enough after the limit $z_0 \rightarrow \infty$ has been taken.

$$\lim_{z_0 \rightarrow \infty} \left(n_0^e - n_0^i \right) = 0 \qquad (3.26)$$

In any layer after z_0, quasineutrality exists.

For f_0^i, (3.23) vanishes identically in the above limit. However a detailed knowledge of f_0^i in z_1 is not needed to obtain the electron current; nevertheless the equation of next order in (3.11) will be considered for this layer in Appendix F. In Section E of this chapter the ions will be studied in the z_2-layer.

The distribution function for the electrons is obtained taking the next higher-order terms in (3.10); although $\frac{\partial f_0^e}{\partial \omega} = 0$ is still valid, this is not the case for the second equation in (3.22). In the limit $z_0 \rightarrow \infty$

we have.

$$\Delta_1^e \frac{\partial f_1^e}{\partial \omega} + \left\{ \delta_1^z V_z \frac{\partial}{\partial Z_1} + \alpha \frac{\partial \phi_0}{\partial Z_1} \frac{\partial}{\partial V_z} \right\} f_0^e$$

$$+ \sigma \left\{ V_\xi \frac{\partial}{\partial \xi_k} + \alpha \frac{\partial \phi_0}{\partial \xi_k} \frac{\partial}{\partial V_\xi} \right\} f_0^e = \gamma^{-1} \varepsilon \left(\frac{\delta f^e}{\delta t} \right)_0 \tag{3.27}$$

where we have written $\delta_k^\xi = \sigma$; also

$$\left(\frac{\delta f^e}{\delta t} \right)_0 \equiv \frac{\delta}{\delta t} \left\{ f_0^e, f_0^e \right\} + \frac{\delta}{\delta t} \left\{ f_0^e, f_0^i \right\}$$

The above expression for $\left(\frac{\delta f^e}{\delta t} \right)_0$ underlines its dependence on the distribution functions. Although the collision operator is not, in general, strictly quadratic in f^e and f^i, it is approximately so. Thus, when f_0^e and f_0^i are small, $\left(\frac{\delta f^e}{\delta t} \right)_0$ is a higher order term. When f_0^e and f_0^i are of order of unity, $\left(\frac{\delta f^e}{\delta t} \right)_0$ is O(1) too.

In general, the terms $\left(\frac{\delta f^e}{\delta t} \right)_j$ are found by sutstituting the expansion (3.9) for f^e and f^i wherever they appear in $\frac{\delta f^e}{\delta t}$ or $\frac{\delta f^i}{\delta t}$. Therefore if use is made of the Balescu-Lenard collision operator there appear some additional terms which are not present in the Fokker-Planck model. However, this does not happen for $\left(\frac{\delta f}{\delta t} \right)_1$ if f_0^e is Maxwellian, as observed at the end of Appendix B.

Eq. (3.27) is of the form

$$\Delta_1^e \frac{\partial f_1^e}{\partial \omega} = A(\omega) + B \tag{3.28}$$

where A and B are independent of f_1^e and B is also independent of ω, i.e.

$$\Delta_1^e f_1^e = \int A(\omega)\, d\omega + B\omega + C \tag{3.29}$$

In order to have f_1^e periodic in ω, B must vanish identically. Eq. (3.28) splits then into

$$\Delta_1^e \frac{\partial f_1^e}{\partial \omega} = A(\omega) \quad , \quad B = 0$$

In (3.27), $A(\omega)$ is given by the terms containing ξ-derivatives; thus we obtain:

$$\Delta_1^e \frac{\partial f_1^e}{\partial \omega} = -\sigma \left\{ V_\xi \frac{\partial}{\partial \xi_k} + \alpha \frac{\partial \phi_0}{\partial \xi_k} \frac{\partial}{\partial V_\xi} \right\} f_0^e$$

In the left-hand side of (3.29), where B vanishes, the first term has a known order of magnitude, σ, but this is not so for the third term, C. Therefore the two parts of $\Delta_1^e f_1^e$ are not related. We can write

$$\Delta_1^e f_1^e \equiv \Delta_{11}^e f_{11}^e + \Delta_{12}^e f_{12}^e = -\sigma \int \left\{ V_\xi \frac{\partial}{\partial \xi_k} + \alpha \frac{\partial \phi_0}{\partial \xi_k} \frac{\partial}{\partial V_\xi} \right\} f^e \, d\omega + \Delta_{12}^e f_{12}^e$$

Thus $\Delta^e_{11} = \sigma$, and f^e_{12} is independent of ω.

Using the identities

$$V_\xi = V_\perp \cos \omega \qquad \frac{\partial f^e_0}{\partial V_\xi} \equiv \frac{\partial f^e_0}{\partial V_\perp}\frac{\partial V_\perp}{\partial V_\xi} + \frac{\partial f^e_0}{\partial \omega}\frac{\partial \omega}{\partial V_\xi} = \frac{\partial f^e_0}{\partial V_\perp}\cos\omega$$

and defining the operator

$$D_\xi \equiv \left\{ V_\perp \frac{\partial}{\partial \xi_k} + \alpha \frac{\partial \phi_0}{\partial \xi_k}\frac{\partial}{\partial V_\perp} \right\}$$

we finally obtain:

$$\Delta^e_1 f^e_1 = -\sigma \sin \omega\, D_\xi f^e_0 + \Delta^e_{12} f^e_{12} \qquad (3.30)$$

The terms independent of ω in (3.27) are those with z-derivatives and the collision term. The first simply because $\frac{\partial f^e_0}{\partial \omega} = 0$. In

$$\left(\frac{\delta f^e}{\delta t}\right)_0 \equiv \frac{\delta}{\delta t}\left\{f^e_0, f^e_0\right\} + \frac{\delta}{\delta t}\left\{f^e_0, f^i_0\right\} \qquad (3.31)$$

the same can be said of the first part (even if the magnetic field is considered in the collision process). While we do not know anything about f^i_0, in Appendix B it is shown that the second part of (3.31) can be expanded in powers of μ:

$$\frac{\delta}{\delta t}\left\{f^e_0, f^i_0\right\} = n^i_0 R(f^e) + O(\mu) \qquad (3.32)$$

$\mathcal{R}$ being a differential operator, independent of ω if its function argument is. The terms $O(\mu)$ are of highest order, of course. Therefore $\left(\frac{\delta f^e}{\delta t}\right)_0$ is independent of ω and we have the equation

$$\delta_1^z\left\{v_z\frac{\partial}{\partial z_1}+\alpha\frac{\partial\phi_0}{\partial z_1}\frac{\partial}{\partial v_z}\right\}f_0^e=\gamma^{-1}\varepsilon\left(\frac{\delta f^e}{\delta t}\right)_0$$

or

$$\delta_1^z=\gamma^{-1}\varepsilon \qquad L_1^z=\lambda$$

$$D_{z_1}f_0^e=\left(\frac{\delta f^e}{\delta t}\right)_0 \tag{3.33}$$

where

$$D_{z_\ell}\equiv\left\{v_z\frac{\partial}{\partial z_\ell}+\alpha\frac{\partial\phi_0}{\partial z_\ell}\frac{\partial}{\partial v_z}\right\}$$

Simple results for the moments of f_1^e can be obtained from (3.30) and (3.33):

$$n^e u_y^{e1}=\frac{-1}{2}\frac{\partial}{\partial\xi_k}\int f_0^e v_\perp^2\,d\vec{v}+n^e\alpha\frac{\partial\phi_0}{\partial\xi_k}$$

$$u_\xi^{e1}=0 \qquad n^e u_z^{e1}=\frac{\Delta_{12}^e}{\sigma}\int f_{12}^e v_z\,d\vec{v} \tag{3.34}$$

$$\frac{\partial}{\partial z_1}n_0^e u_z^{e0}=0$$

Before proceeding to the z_2-layer, some important results will be obtained; they make it possible to match the behavior at large z_1 and small z_2; (in the limit $(\varepsilon,\sigma,\mu)\rightarrow 0$, this corresponds to $z_1\rightarrow\infty$ and $z_2\rightarrow 0$).

If the limit $z_1 \rightarrow \infty$ is taken, (3.30) does not experience any change; however (3.33) becomes

$$0 = \left(\frac{\delta f^e}{\delta t}\right)_0 = \frac{\delta}{\delta t}\{f_0^e, f_0^e\} + m_0^i R(f_0^e) \qquad (3.35)$$

After integrating over $v_z d\vec{v}$, the first term drops out and thus

$$0 = \int R(f_0^e) v_z d\vec{v} \sim m_0^e u_z^{e0}$$

Therefore $\vec{u}_0^e = 0$ (we knew already that $u_\xi^{e0} = u_y^{e0} = 0$). This is a particular aspect of a more general condition of isotropy on f_0^e; the first term in (3.35) forces f_0^e to be Maxwellian; because $\vec{u}_0^e$ is zero the second term vanishes as long as the electron kinetic temperatures along and across $\vec{B}$ are equal. Thus the solution to (3.35) is

$$\lim_{z_1 \to \infty} f_0^e = f_M^e = \frac{m_0^e}{(2\pi)^{3/2}} e^{-\frac{v^2}{2}} \qquad (3.36)$$

where f_M^e is a local, isotropic Maxwellian distribution.

Because the velocity has been normalized with respect to the temperature at infinity, (3.36) implies the assumption that the electron temperature is uniform in z_2 and equal to its value at infinity; this will be proved in Section III-F. Thus the only unknown in f_0^e is its first moment, the density. It should be observed that f_M^e would <u>not</u> be the solution of (3.35), if terms $O(\mu)$ were significant, for an arbitrary f_0^i (*).

From this result it is possible to show that the density in z_0 and z_1 is of order higher than unity, i.e., it goes to zero as $(\varepsilon, \sigma, \mu) \rightarrow 0$. From Eqs. (3.25) and (3.34) we see that in both z_0 and z_1

(*) A point worth emphasizing is that the expansion (3.32) is based on the assumption that the dimensional average velocity of the ions is much smaller than that of the electrons; otherwise it fails. Therefore, if the ions had acquired an average z-velocity of order $\left(\frac{kT_e}{m_e}\right)^{1/2}$ it would not be possible to conclude that $u_z^{e0} = 0$ in the layer following z_1.

However, this alternative has to be rejected here: it would require a gain in energy (non-thermal) for the ions of order

$$m_i \left[\left(\frac{kT_e}{m_e}\right)^{1/2}\right]^2 = \frac{kT_e}{\mu^2}\frac{Z_i}{\beta}$$

μ is very small and this energy gain is enormous; there is no source for it: α is not $O(\mu^{-2})$ but $O(1)$. No similarity exists with Bahm's criterion [5]; this requires an energy gain, for the ions, of order

$$m_i \left[\left(\frac{kT_e}{m_i}\right)^{1/2}\right]^2 = kT_e$$

and moreover, it applies when $T_i \ll T_e$ and α is large enough to repel nearly all the electrons. Then a fraction of αkT_e

$$\frac{1}{\alpha}\alpha kT_e = kT_e$$

accelerates the ions outside the sheath.

$$\frac{\partial}{\partial z_0} n_0^e u_z^{eo} = 0$$

$$\frac{\partial}{\partial z_1} n_0^e u_z^{eo} = 0$$

The flux is not O(1) at $z_1 \to \infty$; it is also not so at $z_0 = 0$. Now if the electron kinetic energy of the motion along $\vec{B}$ is O(1) or larger at $z_0 = 0$, $u_z^{e0} = O(1)$ at least, at $z_0 = 0$ because all electrons go toward the probe. Therefore $n_0^e(z_0 = 0) \neq O(1)$. This result is valid of course only for $\xi_K < 1$ since for $\xi_K > 1$, $u_z^{e0}(z_0 = 0) = 0$ by symmetry. The assumption of the electron energy being O(1) at $z_0 = 0$ is justified on the basis that if it were very small a repelling field would be required and this decreases the density in general but not the average energy.

Now if n_0^e is very small at $z_0 = 0$, it is possible to demonstrate that it is so at $z_1 \to \infty$. From (3.22) we get, integrating over $v_z d\vec{v}$,

$$0 = -\frac{1}{n_0^e}\frac{\partial P_{zz}^{eo}}{\partial z_0} + \alpha\frac{\partial \phi_0}{\partial z_0} \qquad (3.37a)$$

and from (3.33) similarly

$$0 = -\frac{-1}{n_0^e}\frac{\partial P_{zz}^{eo}}{\partial z_1} + \alpha\frac{\partial \phi_0}{\partial z_1} \qquad (3.37b)$$

The collisions do not produce any transfer of momentum between ions and electrons since $n_0^e u_z^{e0}$ is not $O(1)$.

In (3.37a) and (3.37b), $P_{zz}^{e0} \equiv \int f_0^e (v_z - u_z^{e0})^2 \, d\vec{v}$ as was defined in (2.14). Therefore

$$P_{zz}^{e0} = n_0^e \overline{(v_z - u_z^{e0})^2} = n_0^e E_z^{e0}$$

where the bar means average over the distribution function f_0^e; E_z^{e0}, the energy along $\vec{B}$, has been indicated above to be $O(1)$. Then integration of (3.37) is immediate:

$$E_z^{e0} \frac{\partial \ln n_0^e}{\partial z_\ell} + \frac{\partial E_z^{e0}}{\partial z_\ell} = \alpha \frac{\partial \phi_0}{\partial z_\ell} \qquad (\ell = 0, 1)$$

or

$$\ln n_0^e + \ln E_z^{e0} - \alpha \int \frac{\partial \phi_0}{\partial z_\ell} \frac{dz_\ell}{E_z^{e0}} = \text{constant} \quad (3.38)$$

If E_z^{e0} is $O(1)$ and α and ϕ_0 are not large, n_0^e being small at z_0 ($\ln n_0^e$ large) implies that it is so in both layers z_0, z_1, and in particular at $z_1 \to \infty$.

This interesting result, the depletion of the plasma inside the "shadow", will be discussed and illuminated in Sections III-F and III-G.

D3. The z_2-layer

We now let $z_1 \to \infty$ and explore the following (z_2) region which will be seen to be the last one along $\vec{B}$; as $z_2 \to \infty$, the disturbance produced by the probe dies away. Also, the ξ_K-scale is the largest scale of relevance along the ξ-axis.

As was observed in Section D-2, Eq. (3.30) remains unchanged as $z_1 \to \infty$ and Eq. (3.33) reduces to $f_0^e = f_M^e$. To obtain more information about f_0^e in the z_2 layer, we must consider the terms of next order in (3.10), which, for $\xi_{K-1} \to \infty$, $z_1 \to \infty$, are:

$$\Delta_2^e \frac{\partial f_2^e}{\partial \omega} + \delta_2^z \left\{ V_z \frac{\partial}{\partial z_2} + \alpha \frac{\partial \phi_0}{\partial z_2} \frac{\partial}{\partial V_z} \right\} f_0^e + \sigma \alpha \Delta_1^\phi \frac{\partial \phi_1}{\partial \xi_K} \frac{\partial f_0^e}{\partial V_\xi}$$
$$+ \delta_{K+1}^\xi \left\{ V_\xi \frac{\partial}{\partial \xi_{K+1}} + \alpha \frac{\partial \phi_0}{\partial \xi_{K+1}} \frac{\partial}{\partial V_\xi} \right\} f_0^e \qquad (3.39)$$
$$+ \sigma \left\{ V_\xi \frac{\partial}{\partial \xi_K} + \alpha \frac{\partial \phi_0}{\partial \xi_K} \frac{\partial}{\partial V_\xi} \right\} \Delta_1^e f_1^e - s\sigma \frac{V_\eta}{\xi_K} \Delta_1^e \frac{\partial f_1^e}{\partial \omega} = \gamma^{-1} \varepsilon \left(\frac{\delta f^e}{\delta t} \right)_1$$

where $\delta_K^\xi = \sigma$ has been used. Eq. (3.39) is of the form given by (3.28) and a splitting of the equation can be made as in the z_1-layer. An identification of the terms which depend on ω and of those which do not (by substituting $v_\perp$, ω for v_ξ and v_η in (3.39)) results in the two equations

$$\Delta_2^e \frac{\partial f_2^e}{\partial \omega} = -\cos\omega \; \sigma \alpha \Delta_1^\phi \frac{\partial \phi_1}{\partial \xi_K} \frac{\partial f_0^e}{\partial V_\perp} + \delta_{K+1}^\xi \left\{ V_\perp \frac{\partial}{\partial \xi_{K+1}} \right.$$
$$\left. + \alpha \frac{\partial \phi_0}{\partial \xi_{K+1}} \frac{\partial}{\partial V_\perp} \right\} f_0^e + \sigma \left\{ V_\perp \frac{\partial}{\partial \xi_K} \right. \qquad (3.40)$$
$$\left. + \alpha \frac{\partial \phi_0}{\partial \xi_K} \left(\frac{\partial}{\partial V_\perp} - \frac{\tan\omega}{V_\perp} \frac{\partial}{\partial \omega} \right) \right\} \left\{ \Delta_{12}^e f_{12}^e - \sigma \sin\omega \right.$$
$$\left. \left(V_\perp \frac{\partial}{\partial \xi_K} + \alpha \frac{\partial \phi_0}{\partial \xi_K} \frac{\partial}{\partial V_\perp} \right) f_0^e \right\} + s\sigma^2 \frac{V_\perp}{\xi_K} \sin\omega$$

$$\times\left\{V_\perp \frac{\partial}{\partial \xi_k} + \alpha \frac{\partial \phi_0}{\partial \xi_k}\frac{\partial}{\partial V_\perp}\right\} f_0^e + \gamma^{-1}\varepsilon\left(\frac{\delta f^e}{\delta t}\right)_{1\omega}$$

$$\delta_2^z\left\{V_z \frac{\partial}{\partial z_2} + \alpha \frac{\partial \phi_0}{\partial z_2}\frac{\partial}{\partial V_z}\right\} f_0^e = \gamma^{-1}\varepsilon\left(\frac{\delta f^e}{\delta t}\right)_{1c} \qquad (3.41)$$

where $\left(\frac{\delta f^e}{\delta t}\right)_{1\omega}$ and $\left(\frac{\delta f^e}{\delta t}\right)_{1c}$ are the parts of $\left(\frac{\delta f^e}{\delta t}\right)_1$, dependent upon and independent of ω. As we shall see in the next section, f_0^i is Maxwellian in this z_2-layer. Therefore, the terms $O(\mu)$ in (3.32) vanish identically and $\left(\frac{\delta f^e}{\delta t}\right)_1$ is the result of linearizing the collision operator for the electrons:

$$\left(\frac{\delta f^e}{\delta t}\right)_1 \equiv \frac{\delta}{\delta t}\left\{f_0^e, \Delta_1^e f_1^e\right\}_L + \frac{\delta}{\delta t}\left\{\Delta_1^e f_1^e, f_0^e\right\}_L + \frac{\delta}{\delta t}\left\{f_0^e,\right.$$

$$\left.\Delta_1^i f_1^i\right\}_L + \frac{\delta}{\delta t}\left\{\Delta_1^i f_1^i, f_0^e\right\}_L \qquad (3.42)$$

where use of the index L underlines the linearity of the operator in $\Delta_1^e f_1^e, \Delta_1^i f_1^i$.

Eq. (3.42) can be written (using the known form of f_1^e given by (3.30) and the expansion (3.32)):

$$\left(\frac{\delta f^e}{\delta t}\right)_1 \equiv \left[\Delta_{12}^e \frac{\delta}{\delta t}\left\{f_0^e, f_{12}^e\right\}_L + \Delta_{12}^e \frac{\delta}{\delta t}\left\{f_{12}^e\, f_0^e\right\}_L\right.$$

$$+\Delta^i\left\{m_0^i R_L(f_0^e) + O(\mu)\right\} + \Delta_{12}^e\left\{m_0^i R_L(f_{12}^e) +\right.$$

$$\left.\left. + O(\mu)\right\}\right] + \left[-\sigma \frac{\delta}{\delta t}\left\{f_0^e, \sin\omega\, D_\xi(f_0^e)\right\}_L - \sigma\frac{\delta}{\delta t}\right.$$

$$\times\left\{\sin\omega\, D_\xi(f_0^e), f_0^e\right\} - \sigma\left\{m_0^i R_L\left(\sin\omega\, D_\xi(f_0^e)\right)\right.$$

$$\left.\left. + O(\mu)\right\}\right] \equiv \left(\frac{\delta f^e}{\delta t}\right)_{1c} + \left(\frac{\delta f^e}{\delta t}\right)_{1\omega}$$

The terms $O(\mu)$ have been dropped; also $R_L(f_0^e) = 0$ since f_0^e is completely isotropic. Thus we have

$$\left(\frac{\delta f^e}{\delta t}\right)_{1c} = \Delta_{12}^e\left[\frac{\delta}{\delta t}\left\{f_0^e, f_{12}^e\right\}_L + \frac{\delta}{\delta t}\left\{f_{12}^e, f_0^e\right\}_L + \dot{m_0} R_L(f_{12}^e)\right] \quad (3.43)$$

$$\left(\frac{\delta f^e}{\delta t}\right)_{1\omega} = -\sigma\left[\frac{\delta}{\delta t}\left\{f^e, \sin\omega\, D_\xi(f_0^e)\right\}_L + \frac{\delta}{\delta t}\left\{\sin\omega\, D_\xi(f_0^e),\right.\right.$$

$$\left.\left. f_0^e\right\}_L + \dot{m_0} R_L\left(\sin\omega\, D(f_0^e)\right)\right] \quad (3.44)$$

Then, in (3.41), we find

$$\delta_2^{-z} = \gamma^{-1}\varepsilon\,\Delta_{12}^e \quad (3.45a)$$

$$D_{z_2}(f_0^e) = \frac{\delta}{\delta t}\left\{f_0^e, f_{12}^e\right\}_L + \frac{\delta}{\delta t}\left\{f_{12}^e, f_0^e\right\}_L + \dot{m_0} R_L(f_{12}^e) \quad (3.45b)$$

This equation will be reconsidered in the next section.

Substitution of (3.44) into (3.40) allows integration; however, the complete form for f_2^e is not needed but only that part of it which gives a non-vanishing u_ξ^{e2}; (we have seen in the two previous sections that $u_\xi^{e0} = u_\xi^{e1} = 0$). The terms in (3.40) that contribute to u_ξ^{e2} are easily found. By definition,

$$\Delta_2^e n_\xi^e u_\xi^{e2} = \Delta_2^e \int f_2^e V_\xi\, d\vec{V} = \Delta_2^e \int_{-\infty}^{\infty} dV_z \int_0^{\infty} V_\perp dV_\perp \int_0^{2\pi} f_2^e$$

$$\times V_\perp \cos\omega\, d\omega$$

Integrating by parts with respect to ω, there results

$$\Delta_2^e n_2^e u_\xi^{e2} = -\Delta_2^e \int \frac{\partial f_2^e}{\partial \omega} v_y \, d\vec{v}$$

Since $\int_0^{2\pi} \cos \omega \sin \omega \, d\omega = 0$, and $\int_0^{2\pi} \sin 2\omega \sin \omega \, d\omega = 0$ only $\left(\frac{\delta f^e}{\delta t}\right)_{1\omega}$ can contribute to u_ξ^{e2}. Therefore:

$$\Delta_2^e n_2^e u_\xi^{e2} = -\gamma^{-1} \varepsilon \int \left(\frac{\delta f^e}{\delta t}\right)_{1\omega} v_y \, d\vec{v}$$

or

$$\Delta_2^e = \gamma^{-1} \varepsilon \sigma = \frac{\ell_e^2}{\lambda R} \tag{3.46a}$$

$$n_2^e u^{e2} = \int v_y \, d\vec{v} \left[\frac{\delta}{\delta t} \left\{ f_0^e, \sin \omega \, D(f_0^e) \right\}_L + \frac{\delta}{\delta t} \left\{ \sin \omega \times D(f_0^e), f_0^e \right\}_L + n_0^{i} R_L \left(\sin \omega \, D_\xi(f_0^e) \right) \right] \tag{3.46b}$$

Thus the order of magnitude of the transverse flux, except possibly in a region around $\xi = R$ shrinking to zero as $(\varepsilon, \sigma, \mu) \to 0$, is of order of magnitude (in dimensional variables) $N_\infty \left(\frac{kT_e}{m_e}\right)^{1/2} \frac{\ell_e^2}{\lambda R}$; this collisional diffusion is proportional to B^{-2} because $\ell_e \sim B^{-1}$.

To obtain $\bar{\delta}_z^2$ (and Δ_{12}^e) in (3.44) and thereby determine the magnitude of the flux to the probe and the z_2-characteristic length, we need only to establish the balance of fluxes through the continuity equation. Since $n_0^e u_z^{e0} = 0$, we need: $\Delta_1^e n_1^e u_z^{e1} \equiv \Delta_{12}^e \int f_{12} v_z \, d\vec{v}$. From (3.30) and (3.46), the continuity equation is, in the limit

$(z_1, \xi_{K-1}) \rightarrow \infty$:

$$\delta_2^z \frac{\partial}{\partial z_2} \Delta_{12}^e n_1^e u_z^{e1} + \frac{1}{\xi_k^s} \delta_k^\xi \frac{\partial}{\partial \xi_k} \xi_k^s \Delta_2^e n_2^e u_\xi^{e2} = 0 \quad (3.47)$$

or

$$\Delta_{12}^e \delta_2^z = \delta_k^\xi \Delta_2^e \quad (3.48a)$$

$$\frac{\partial}{\partial z_2} n_1^e u_z^{e1} + \frac{1}{\xi_k^s} \frac{\partial}{\partial \xi_k} \xi_k^s n_2^e u_\xi^{e2} = 0 \quad (3.48b)$$

Use of (3.45a) and (3.46a) and also $\delta_k^\xi = \sigma$, gives:

$$(\delta_2^z)^2 \varepsilon^{-1} \gamma = \gamma^{-1} \varepsilon \sigma^2, \quad \text{or} \quad \delta_2^z = \sigma \varepsilon \gamma^{-1} \quad (3.49a)$$

Therefore

$$L_2^z = \frac{\lambda R}{\ell_e}, \quad \Delta_{12}^e = \sigma \quad (3.49b)$$

This balance of fluxes allows the disturbances produced by the probe to die out as $z_2 \rightarrow \infty$ in the ξ_k-region; the space perturbed by the probe is thus of order $\frac{\lambda R}{\ell_e}$ along $\vec{B}$ and R across $\vec{B}$. The current to the probe decreases with respect to the value for $B = 0$, by a factor of order $\sigma = \frac{\ell_e}{R}$.

E. The complete description in the z_2-layer

E1. The closure of the equations

To solve (3.48b), $n_1^e u_z^{e1}$ and $n_2^e u_\xi^{e2}$ are obtained from (3.45b) and (3.46b). The only unknown in f_0^e is n_0^e, and $n_0^i = n_0^e$ from (3.26); then (3.48b) gives a relation between n_0^e and ϕ_0. The other relation needed does not come from Poisson's equation; (it has been used to find $n_0^e = n_0^i$ and higher terms would involve n_1^e, n_1^i). Instead, the equation for f^i is used now; this, and again quasineutrality, provide a closure to the problem in the z_2-layer. The relation we shall find is simply $n_0^e = n_0^i = e^{-\frac{\alpha}{\beta}\phi_0}$.

After the limits $\xi_{K-1} \rightarrow \infty$, $z_1 \rightarrow \infty$ have been taken in (3.11) the dominant terms from every summation are:

$$\delta_2^z \left\{ V_z \frac{\partial}{\partial z_2} - \frac{\alpha}{\beta} \frac{\partial \phi_0}{\partial z_2} \frac{\partial}{\partial V_z} \right\} f_0^i + \delta_K^\xi \left\{ V_\xi \frac{\partial}{\partial \xi_K} - \frac{\alpha}{\beta} \frac{\partial \phi_0}{\partial \xi_K} \frac{\partial}{\partial V_\xi} \right\}$$

$$\times f_0^i - s\,\delta_K^\xi \frac{V_\eta}{\xi_K} \frac{\partial f_0^i}{\partial w} - \mu \frac{\partial f_0^i}{\partial w} = \gamma^{-1} \varepsilon \left(\frac{\delta f^e}{\delta t} \right)_0 \qquad (3.50)$$

Because $\delta_2^z = \gamma^{-1} \varepsilon \sigma$, the first term is of higher order than all the others with the possible exception of the last one in the left-hand side. Assume first the case $s = 0$ and $\mu \ll \gamma^{-1} \varepsilon$. Eq. (3.50) becomes

$$\sigma\left\{V_\xi \frac{\partial}{\partial \xi_k} - \frac{\alpha}{\beta}\frac{\partial \phi_0}{\partial \xi_k}\frac{\partial}{\partial V_\xi}\right\} f_0^i = \gamma^{-1}\varepsilon \left.\frac{\delta f^e}{\delta t}\right|_{\circledast} \quad (3.51)$$

From now on, in order to be able to proceed rigorously, the case $\lambda \ll R$ will be dropped from our study; anyhow for a fully ionized plasma the above inequality is extremely infrequent. Then, in a ξ_j-scale, $j < k$, one has only to drop the right-hand side of (3.51) and change the index j for k in the ξ-derivatives; (see (3.52) below). If $\lambda \gg R$ (that is, $\gamma^{-1}\varepsilon \equiv \frac{l_e}{\lambda} \ll \frac{l_e}{R} \equiv \sigma$), the collision term is dropped also in the ξ_k-scale, but this assumption is not needed.

Now from (3.51) and

$$\delta_j^\xi \left\{V_\xi \frac{\partial}{\partial \xi} - \frac{\alpha}{\beta}\frac{\partial \phi_0}{\partial \xi_j}\frac{\partial}{\partial V_\xi}\right\} f_0^i = 0 \quad (3.52)$$

valid for $j < k$ and $z_1 \rightarrow \infty$, f_0^i is seen to be a Maxwellian distribution with a temperature identical to that at infinity and $u_\xi^{i0} = 0$. This result arises from: a) f_0^e is Maxwellian in the ξ_k-region, where the collision term appears in (3.51); b) ϕ_0 is a potential field; c) at infinity f_0^i becomes Maxwellian; and d) at $\xi_k = 0$, perfect "reflection" exists because of symmetry. Conditions or effects along the z-axis are of no consequence; z-gradients are too weak and absorbing boundary conditions are immaterial far from the probe.(*)

Then from (3.51) and (3.52) we obtain

$$n_0^i = A(z_2)\, e^{-\frac{\alpha}{\beta}\phi_0} \qquad (3.53)$$

$A(z_2)$ being an unknown function. But we have

$$\lim_{\xi_k \to \infty} \phi_0 = 0$$

and

$$\lim_{\xi_k \to \infty} n_0^i = 1$$

there results $A(z_2) \equiv 1$.

We obtain for any ξ_j-layer with $L_j^{\xi} \gg \lambda_D$, from quasineutrality:

$$n_0^e = n_0^i = e^{-\frac{\alpha}{\beta}\phi_0} \qquad (3.54)$$

We shall see in Appendix F that ξ-gradients around $\xi = R$ have been smoothed out in the z_1-layer;

(*) If, $\lambda \ll R$, equation (3.54) would give a local Maxwellian distribution for f_0^1.

in fact L_0^{ξ} in the z_2-layer is larger than ℓ_e and λ_D. Therefore, both the formulation we have found in the preceding section for f_0^e and quasineutrality are valid throughout the z_2-layer.

In this layer, thus, we have a well-defined problem which is the solution of (3.48b), using (3.37), (3.45b), (3.46b) and (3.54).

Going back to (3.50), if we now allow the fourth term to be retained (this is only necessary if $\frac{\mu}{\sigma} = O(1)$) the result for f_0^i should not be modified by the presence of a magnetic field. This means that the condition $\mu \ll \sigma$ (or $\ell_i \gg R$) is not necessary, since it has not been used at all, to obtain the electron current to the probe. Our results will be valid as long as ℓ_i is not much smaller than R. However, in Chapter IV it will be seen that to obtain I^i $\ell_i > R$ should at least be satisfied; as commented in Section II-B the stronger condition $\ell_i \gg R$ seems not necessary.

In addition, the inclusion of the third term in (3.50), (s = 1) will not modify the equilibrium of f_0^i because this equilibrium is independent of the coordinate system used.

E2. The equation for the electric field

Use of (3.37) in (3.45b) reduces the left-hand side to the form $\frac{\beta+1}{\beta}\,\alpha\,\frac{\partial\phi_0}{\partial z_2}\frac{\partial f_0^e}{\partial v_z}$. The equation can be written in dimensional variables:

$$\frac{e}{m_e}\,\frac{\beta+1}{\beta}\,\frac{\partial V}{\partial z}\,\frac{\partial F_0^e}{\partial W_z} = \left(\frac{\delta F^e}{\delta t}\right)_1 \qquad (3.55)$$

and is the equation for the electric conductivity along a magnetic field; the factor $\frac{\beta+1}{\beta}$ takes into account that, because of the Maxwellian form of f_0^e, both terms in $D_{z_2}(f_0^e)$ are similar and the diffusion can be incorporated into the conductivity.

Eq. (3.55) has been solved by many authors when the magnetic field does not enter the collision process; in this case $\vec{B}$ has no influence on the result. We quote from Appendix E, Eq. (E.2) for the flux in non-dimensional form, in this case:

$$n_i^e u^{e1} = \Big[N_{\infty}\,\lambda_i^2\,\lambda\,\ln\Lambda(\xi_k,z_2)\left(1+\frac{0.245}{\ln\Lambda(\xi_k,z_2)}\,\frac{2^{5/2}}{\pi^{3/2}}\right)\Big]^{-1} \times \frac{\gamma_E(z_i)}{z_i}\,\frac{\beta+1}{\beta}\,\alpha\,\frac{\partial\phi_0}{\partial z_2} \qquad (3.56)$$

The possible extension to the case $\lambda_D \gg \ell_e$ is discussed in Appendix E. In (3.56), Λ has a local value. The function $\delta_E(z_i)$ is given by Spitzer [52] and goes from 0.582 for $z_i = 1$ to 1 for $z_i = \infty$.

Similarly, using (3.37) in (3.46b) and observing that the collision operator is linear in $D_\xi(f_0^e)$, we can write the last equation as:

$$n_2^e u_\xi^{e2} = \frac{\beta+1}{(2\pi)^{3/2}} \frac{\partial n_0^e}{\partial \xi_k}$$
$$\times \int v_\eta \, d\vec{v} \left[\frac{\delta}{\delta t} \left\{ f_0^e, v_\eta e^{-v^2/2} \right\}_L + \frac{\delta}{\delta t} \right.$$
$$\left. \times \left\{ v_\eta e^{-v^2/2}, f_0^e \right\}_L + n_0^i R_L \left(v_\eta e^{-v^2/2} \right) \right]$$

This equation serves to determine the diffusion across a large magnetic field. The result of its solution is quoted from (E.13)

$$n_2^e u_\xi^{e2} = \left(\frac{2}{\pi}\right)^{1/2} \frac{4\pi}{3} \frac{\beta+1}{\beta} Z_i N_\infty \lambda_L^2 \lambda$$
$$\times \left(\ln \Lambda(\xi_k, z_i) - 0.5 \right) \alpha \frac{\partial \phi_0}{\partial \xi_k} e^{-\frac{2\alpha}{\beta}\phi_0} \tag{3.57}$$

This result is valid for $\lambda_D < \ell_e$; the extension to the case $\lambda_D > \ell_e$ is discussed in Appendix E.

Making use of (3.56) and (3.57), Eq. (3.47) becomes

$$\frac{\gamma_E}{\pi Z_i}\left(N_\infty \lambda_L^2 \lambda\right)^{-1}\frac{\partial}{\partial z_2}\left(\ln\Lambda + 0.245\right)^{-1}\frac{\partial \phi_0}{\partial z_2}$$
$$+\frac{\pi}{3}\; N_\infty \lambda_L^2 \lambda^{'}\,\frac{1}{\xi_k^s}\frac{\partial}{\partial \xi_k}\xi_k^s\left(\ln\Lambda - 0.5\right)e^{-\frac{2\alpha}{\beta}\phi_0} \qquad (3.58)$$
$$\times\frac{\partial \phi_0}{\partial \xi_k} = 0$$

Although the temperature is uniform in this layer, the density is not; thus $\ln\Lambda$ depends on z_2, ξ_k in the following way:

$$\ln\Lambda = \ln\Lambda_\infty - \ln\left(n_0^e\right)^{1/2} = \ln\Lambda_\infty - \frac{\alpha\phi_0}{2\beta} \qquad (3.59)$$

where

$$\Lambda_\infty = \frac{3\lambda_D}{\lambda_L} = \frac{3(KT_e)^{3/2}}{e^3\left(4\pi N_\infty\right)^{1/2}}$$

Therefore there is a weak variation in the transport coefficients throughout the layer, represented by (3.59), and a strong one from the exponential $e^{-4\left(\frac{\alpha\phi_0}{2\beta}\right)}$.

Although Eq. (3.58) is not untractable, we shall simplify it, from now on, by neglecting the weaker dependence on Λ cited above. The basic reason to do this is a difficulty which is extrinsic to the present problem and which is the imperfect state of the knowledge of transport coefficients in the presence of a strong magnetic field. In this respect, assume that $\lambda_D < \ell_e$. This inequality is satisfied at infinity; for values of z_2 near zero and $\xi_K < 1$, the local Debye length is much larger than its value at infinity (λ_D), while the local electron Larmor radius is the same because it depends only on the temperature. Thus, it is possible that for some region of the $z_2 - \xi_K$ plane $\lambda_D > \ell_e$; then the expression for $n_1^e u_z^{e1}$ and $n_2^e u_\xi^{e2}$ given by (3.56) and (3.57) would not be correct. At present no formulas are available for these transport coefficients for the whole range $0 < \ell_e < \infty$ (see Appendix E).

In any event, the dominant effect is contained in the exponential in (3.58); moreover for $\Lambda_\infty \to \infty$ the relative change in $\ln \Lambda$ goes to zero. We shall neglect this variation by choosing an average value for $\ln \Lambda$; an important result obtained below is that the current does not depend on this average value.

With this assumption we write from (3.58)

$$\frac{\gamma_E}{\pi Z_i}\left(N_\infty \lambda_L^2 \lambda \ln \bar{\Lambda}\right)^{-1} \frac{\partial^2 \phi_0}{\partial z_2^2} + \frac{\pi}{3} N_\infty \lambda_L^2 \lambda \ln \bar{\Lambda} \times \frac{1}{\xi_K^5}\frac{\partial}{\partial \xi_K} \xi_K^5 \, e^{-\frac{2\alpha\phi_0}{\beta}} \frac{\partial \phi_0}{\partial \xi_K} = 0$$

Defining

$$-\psi = \frac{2\alpha}{\beta}\phi_0$$

$$\zeta = z_2 \frac{\pi Z_i N_\infty \lambda_L^2 \lambda \ln \bar{\Lambda}}{(3\gamma_E)^{1/2}} \qquad (3.61)$$

this equation becomes

$$\frac{\partial^2 \psi}{\partial \zeta^2} + \frac{1}{\xi_k^s}\frac{\partial}{\partial \xi_k}\xi_k^s\, e^{-\psi}\frac{\partial \psi}{\partial \xi_k} = 0 \qquad (3.62)$$

and (3.57) becomes

$$m_i u_z^{e\prime\prime} = \left(N_\infty \lambda_L^2 \lambda \ln \bar{\Lambda}\right)^{-1} \frac{2^{5/2}}{\pi^{3/2}}\frac{\gamma_E}{Z_i}\frac{\beta+1}{2}$$

$$(3.63)$$

$$\times \frac{\partial \psi}{\partial \zeta}\frac{\pi Z_i N_\infty \lambda_L^2 \lambda \ln \bar{\Lambda}}{(3\gamma_E)^{1/2}} = \frac{2^{3/2}}{\pi^{1/2}}\left(\frac{\gamma_E}{3}\right)^{1/2}(\beta+1)\frac{\partial \psi}{\partial \zeta}$$

Eqs. (3.62) and (3.63) are both independent of $\bar{\Lambda}$.

To find the current to the probe we need (3.63) at $z_2 = 0$, $\xi_k < 1$. Therefore it is necessary to solve (3.62). The boundary conditions for this non-linear elliptic equation are:

$$\lim_{(\xi_k^2+\zeta^2)^{1/2}\to\infty} \psi = 0 \qquad (3.64a)$$

$$\left.\begin{aligned} \frac{\partial\psi}{\partial\zeta} &= 0 \quad \text{at} \quad \zeta = 0,\ \xi_K > 1 \\ h\left(\psi, \frac{\partial\psi}{\partial\zeta}\right) &= 0 \quad \text{at} \quad \zeta = 0,\ \xi_K < 1 \end{aligned}\right\} \quad (3.64b)$$

and for $s = 0$

$$\frac{\partial\psi}{\partial\xi_K} = 0 \quad \text{at} \quad \xi_K = 0 \qquad (3.64c)$$

Eq. (3.64a) expresses the vanishing of the potential at infinity. (3.64b) arises from the conservation of z-flux in the z_0, z_1-layers; at $z_0 = 0$ the z-flux is zero for $\xi_K > 1$ by symmetry, and is equal to the current density to the probe for $\xi_K < 1$. Then the first equation in (3.64b) follows from (3.63) and the second, where h is unknown as yet, will be obtained in the next section and expresses the equality of z-fluxes at $z_1 \to \infty$ and $z_2 \to 0$. Condition (3.64c) for $s = 0$, can be eliminated by substituting $|\xi_K|$ for ξ_K in (3.64b).

It should be pointed out that the z-flux is conserved in the ξ_K but not in the $\xi_j (j < K)$-regions. There a smoothing of gradients develops, as one advances from the probe in the z_0, z_1-layers. Therefore no boundary conditions are available in the neighborhood of ($\xi_K = 1$, $z_2 = 0$) until a detailed analysis of the $\xi_j (j < K)$-regions in z_0, z_1 is made. Nevertheless we know that in the limit $(\varepsilon, \sigma, \mu) \to 0$, this zone of indeterminancy shrinks to zero. Numerically, an arbitrariness in the boundary conditions of an elliptic equation on a very small part of the contour has a negligible effect except possibly because of singularities. As we shall see in Chapter IV, not even an integrable singularity appears at $\xi_K = 1$, $\zeta = 0$ and no difficulties are met in either the computations or the analytical results.

E3. The electron temperature

In Section D it was assumed that in the z_2-layer the electron temperature was uniform and equal to its value at infinity. This can be understood by ~~observing that~~ normally the heating of the electrons is neglected when calculating the electrical conductivity; this is correct if the field is very weak. Here $\lambda << \frac{\lambda R}{\ell_e} = L_n^z$ so that many collisions occur before a significant acceleration is produced by the field. An average momentum is gained; if this is very small (and we know that $u_z^e << 1$), the average energy gain is of a higher order.

Moreover, if $\mu = O(\sigma)$, this can be demonstrated explicitly. In effect, when it was asserted that the terms $O(\mu)$ in (3.32) vanish because of f_0^e and f_0^i being Maxwellian, an implicit assumption was made that the local ion and electron temperatures were equal; only then do we have

$$\frac{\delta}{\delta t}\left\{f_M^e, f_M^i\right\} = 0$$

To verify this assumption the terms of $O(\mu)$ must be retained in $\left(\frac{\delta f^e}{\delta t}\right)_1$, if μ is comparable to σ. Because f_M^e and f_M^i are both isotropic they should be added to (3.43)

$$D_{z_2}(f_0^e) = \left(\frac{\delta f^e}{\delta t}\right)_{1c} + \left[O(\mu)\right]_{f_0^e, f_0^i}$$

Integrating over $v^2 d\vec{v}$, the left-hand side vanishes and the first term on the right also.(*) Therefore

$$0 = \int v^2 d\vec{v} \left[O(\mu)\right]_{f_0^e, f_0^i}$$

Thus the electron temperature is equal to that of the ions; in particular the equality is true at infinity.

F. The interior layers

We established in Section D that $n_0^e(z_1 \to \infty)$ was not $O(1)$ but very small. Because $n_0^e(z_2 \to 0) = n_0^e(z_1 \to \infty)$,

(*) The electron self-collisions do not produce heating and as seen in Appendix B, for any f_{12}^e, $\int v^2 d\vec{v}\ R\ (f_{12}^e) d\vec{v} = 0$.

$n_0^e = e^{-\frac{\alpha}{\beta}\phi_0}$ and α and β are $O(1)$, in the z_2-layer $\phi_0 \neq O(1)$; it should be very large as $z_2 \to 0$.

That $\phi_0 = O(1)$ as $z_2 \to 0$ may be observed from the analysis for the z_0, z_1-layers in Section D; the first-order z-flux is conserved from the probe to the base of the z_2-layer, but $u_z^{e0} = O(1)$ at $z_0 = 0$ and of higher order at $z_1 \to \infty$. Therefore n_0^e must be much larger at $z_1 \to \infty$ than at $z_0 = 0$. From the argument given in Section D2, Eq. (3.38), this is not possible unless ϕ_0 is very large.

Therefore, the order of V inside the "shadow" has no relation to V_p. To split the normalized potential $\frac{eV}{kT_e}$ in the form

$$\frac{eV}{kT} = \frac{eV_p}{kT} \; \frac{V}{V_p} = \alpha\phi_0$$

has meaning no longer; therefore, we shall write

$$\chi = \frac{eV}{kT_e} \tag{3.65}$$

so that $\chi_p \equiv \alpha$.

Using χ_0 for $\alpha\phi_0$, the equations for f_0^e in z_0 and z_1 are written, from (3.22) and (3.33):

$$v_z \frac{\partial f_0^e}{\partial z_0} + \frac{\partial \chi_0}{\partial z_0} \frac{\partial f_0^e}{\partial v_z} = 0 \tag{3.66a}$$

$$v_z \frac{\partial f_0^e}{\partial z_1} + \frac{\partial \chi_0}{\partial z_1} \frac{\partial f_0^e}{\partial v_z} = \left(\frac{\partial f^e}{\partial t}\right)_0 \qquad (3.66b)$$

The first is linear in f_0^e; thus, it is not modified by $n_0^e \ll 1$. In the second, the right-hand side is roughly quadratic in n_0^e. We can drop it therefore and for both z_0 and z_1 f_0^e satisfies(*)

$$v_z \frac{\partial f_0^e}{\partial z_\ell} + \frac{\partial \chi_0}{\partial z_\ell} \frac{\partial f_0^e}{\partial v_z} = 0 \qquad (\ell = 0,1) \qquad (3.66)$$

It should be remembered that the separation in z_0 and z_1 was made using the characteristic lengths λ_D and λ (the values of the undisturbed plasma). Since $\lambda_D \sim (n_0^e)^{-1/2}$, $\lambda \sim (n_0^e)^{-1}$ these layers become blurred.

The solution to (3.66) is

$$f_0^e = g\left(\frac{v_z^2}{2} - \chi_0\right) H\left[2^{1/2}(\chi_0 - \chi_p)^{1/2} - v_z\right] \qquad (3.67)$$

where g is an arbitrary function and H is the step function

$$H(x) = 1, \quad x > 0$$
$$H(x) = 0, \quad x < 0$$

(*) In this section all the equations are restricted to the range $\xi_k < 1$.

Observing that Dirac's δ-function is the "derivative" of H, one can show that f_0^e as given in (3.67) satisfies (3.66).

Now g has to be found using the conditions at $z_1 \to \infty$, that is, at $z_2 \to 0$. There f_0^e should be Maxwellian. If we write in (3.67)

$$g = Ae^{\chi_o} e^{-\frac{V_z^2}{2}} e^{-\frac{V_\perp^2}{2}}$$

where A does not depend on z_0 or z_1, we can expand f_0^e in spherical velocity coordinates using Legendre polynominals, P_ℓ:

$$f_0^e = \sum_\ell P_\ell(\nu)\varphi_\ell(v) \tag{3.68}$$

where

$$\nu = \frac{V_z}{V}$$

Then we find from (3.67) and (3.68)

$$\varphi_0(v) = Ae^{\chi_o} e^{-\frac{V^2}{2}} + \frac{A}{2} e^{\chi_0} e^{-\frac{V^2}{2}}$$
$$\times \left[\frac{2^{1/2}\{\chi_0 - \chi_P\}^{1/2}}{V} - 1\right] H\left[V - 2^{1/2}\{\chi_0 - \chi_P\}^{1/2}\right] \tag{3.69}$$

$$\varphi_1(v) = \frac{3}{4} A e^{\lambda_0} e^{-\frac{v^2}{2}} \left[\frac{2(\lambda_0 - \lambda_P)}{v^2} - 1 \right]$$
$$\times H\left[v - 2^{1/2}\{\lambda_0 - \lambda_P\}^{1/2} \right] \tag{3.70}$$

Since $\lambda_P \equiv \alpha = O(1)$ and $\lambda(z_1 \to \infty) \to \infty$ as $(\varepsilon, \sigma, \mu) \to 0$, the second term in $\varphi_0(v)$ and $\varphi_\ell(\ell \geqq 1)$ are of higher order than the first term in φ_0 because of the absence of the step function.

Now we knew

$$f_0^e(z_1 \to \infty, z_2) = e^{-\frac{1}{\beta}\lambda_0(z_1 \to \infty, z_2)} \frac{e^{-\frac{v^2}{2}}}{(2\pi)^{3/2}}$$

Therefore

$$A(z_2) = \frac{e^{-\frac{\beta+1}{\beta}\lambda_0(\infty, z_2)}}{(2\pi)^{3/2}} \tag{3.71}$$

With this value of A we can find the z-flux in the z_0, z_1 layers using (3.68):

$$n_0^e u_z^{e0} = \int \sum_\ell P_\ell(v) \varphi_\ell(v)\, v\nu\, d\vec{v}$$

and using (3.70) and (3.71) we obtain

$$\int P_1 \varphi_1 v_z\, d\vec{v} = -\frac{e^{\lambda_P} e^{-\frac{\beta+1}{\beta}\lambda_0(z_2=0)}}{(2\pi)^{1/2}} \tag{3.72}$$

since $z_2 = 0$ in the z_0, z_1-layer s.

The order of magnitude of χ_0 at $z_2 = 0$ can be found observing that at $z_0 = 0$, $n_0^e = O(\sigma)$. From (3.69),

$$A(z_2 = 0) = e^{-\frac{\beta+1}{\beta}\chi_0(z_2=0)} = O(\sigma)$$

Therefore

$$\chi_0\,(z_2=0) = O\left(\ln \sigma^{-\frac{\beta}{\beta+1}}\right)$$

In this chapter it has been assumed that $\alpha = O(1)$; when α is taken larger and larger it will be seen in Chapter IV that α finally exceeds $\chi_0(z_2 = 0)$ because $\chi_0(z_2 = 0)$ grows slowly with α.

The formulation given here then fails. This will be considered in Chapter IV. When $\alpha \leq \chi_0$, both terms in (3.69) are of the same order of magnitude; then, the result (3.72) for the current can be extended to these large, positive values of α by matching the density using the complete φ_0, instead of its first term in (3.69) as $z_1 \to \infty$, to $n_0^e(z_2 \to 0)$. Then we obtain for A:

$$A = \frac{2}{1+\operatorname{erf}\{\chi_0 - \chi_p\}^{1/2}} \; \frac{e^{-\frac{\beta+1}{\beta}\chi_0(\infty, z_2)}}{(2\pi)^{3/2}}$$

and for the z-flux

$$\int P_1 \varphi_1 v_z d\vec{v} = -\left[\frac{2}{1+\operatorname{erf}(\chi_0-\chi_p)^{1/2}}\right] \frac{e^{-\frac{\beta+1}{\beta}\chi_0(z_1\to\infty)}\, e^{\chi_p}}{(2\pi)^{1/2}} \tag{3.73}$$

As $\left[\chi_0(z_2 = 0) - \chi_p\right] \to \infty$ the bracket in (3.73) approaches 1 very fast: for $(\chi_0 - \chi_p)$ as small as 1, it differs from unity by about 8%; for $(\chi_0 - \chi_p) > 2$ the error is less than 1%. As α increases with σ fixed, $\left[\chi_0(z_2 = 0) - \chi_p\right]$ approaches zero. Then $\varphi_1(v)$, as given in (3.70), becomes comparable to $\varphi_0(v)$; therefore, at $z_2 = 0$, $\Delta_1^e f_1^e = O(f_0^e)$ and the expansion for f^e fails. Also f^e cannot approach a Maxwellian form at zero order as $z_1 \to \infty$. The value of χ_p for which $\chi_0(z_2 = 0, \chi_p) = \chi_p$ is a "breaking point" in the formulation of this section; the results for $(\chi_0(z_2 = 0) - \chi_p) < 1$ or 2 are not quantitatively reliable. In fact this will be observed in the computations discussed in Chapter IV.

It will be shown in Section IV-A4 that equation (3.63) implies a saturation of I^e; as $\sigma \to 0$, the value of χ_p for which the overshooting disappears moves toward infinity. Thus the electron saturation current can be found within the present formulation. However for the small but non-zero values of σ of interest, the overshooting disappears before saturation has been achieved.

The extension of the present theory to larger α's could be accomplished in the following way. As α increases, f_0^e cannot be Maxwellian as $z_1 \to \infty$ but it is required to be Maxwellian as $z_2 \to 0$. This contradiction in the limits would have to be resolved by using an intermediate layer in which both z-gradients and collisions are taken into account. (Collisions are neglected in z_1 and z-gradients are neglected in z_2.) The equation for this sublayer (the z_2' layer) is

$$v_z \frac{\partial f_0^e}{\partial z_2'} + \frac{\partial \chi_0}{\partial z_2'} \frac{\partial f_0^e}{\partial v_z} = \left(\frac{\delta f^e}{\delta t}\right)_0 \qquad (3.74)$$

and the resulting solution should match the (z_0, z_1) solution as $z_2' \to 0$ and the z_2-solution as $z_2' \to \infty$. This intermediate layer is imbedded in z_2 and exists only for ξ_k 1; the characteristic length $L_{2'}^z$ changes continuously from L_1^z to L_2^z. The solution can be completed (with considerable effort) by recognizing that (3.62) is basically unchanged, (3.54) is correct in z_2', and that any possible change in the transport coefficients arising from $f_0^e = f_M^e$ are unimportant (when $n_0^e \ll 1$, the first term of (3.62) should vanish; see Section IV-A4).

When the overshooting exists $\varphi_1 \ll \varphi_0$ for $z_1 \to \infty$; thus in the whole z_2 layer $\Delta_1^e f_1^e \ll f_0^e$. Moreover, the z_2' sublayer becomes redundant because (3.74) has as solution a global Maxwellian: $f_0^e = f_M^e$ for $z_2' \to \infty$; electrons are strongly repelled for $z_2' \to 0$ (as at a reflective boundary) and the field is a potential one.

Comparing (3.73) to the expression for $\Delta_1^e n_1^e u_z^{e1}$ we found in z_2 gives the relation between ψ and $\frac{\partial\psi}{\partial\zeta}$ represented by the second (3.64b) equation.

G. The coupling of probe and magnetic field

The preceding sections have provided a description of the effects caused by the presence of a strong magnetic field upon the plasma perturbed by the probe.

The first is the loss of importance of Poisson's equation. In fact as we shall see in Section V-A, the present formulation seems to be valid for weaker fields all the way down up to $B = 0$, for some situations in which Poisson's equation is not used.

Second, as was pointed out under certain conditions by Bohm [5], the shape of the probe along $\vec{B}$ has no sensible influence on the collected current. The analysis of this chapter gives a clear explanation: a probe with dimension along $\vec{B}$ of order λ at most, lies entirely in the plane $z_2 = 0$; thereby only its cross section appears in the formulation for the z_2-scale. In the interior z-layers a detailed description of the plasma around the probe is not needed. Therefore, our results will be applicable to a cylinder and a sphere as much as to a strip and a disc.

A third important consideration is the possibility that the global behavior of the plasma affects the collection of the current. Non-classical and non-local transport effects are possibly present unless the plasma is in a very smooth state.

The experimentally observed decrease in the current and blurring of space potential have now a very clear explanation. Basically the following picture emerges: because of the inhibition of the transverse electron flux, any electron current collected by the probe is maintained over long distances along the field. In the absence of enormously large probe potentials the ions are, to first order in μ, motionless with respect to the electrons; any sensible flux would experience a friction with the ions over so many mean free paths that only a small value of electron flux is possible.(*) In fact the value of the electron flux and the extent of the probe perturbation were found by way of a balancing of these effects together with the idea that the magnitude of a gradient is determined by the size of the region in which it exists.

(*) This implies a low value of density near the probe.

This explains the observed decrease in the electron current. But a new and decisive effect appears simultaneously and explains the "blurred" chatacter of the space potential. Finite distances in z_2 require, through Poisson's equation, quasineutrality. Because ions are not inhibited in flowing across the field, only large differences in electric potential can produce large differences in ion density between the inside and outside of the "shadow". The result is that a large electric potential is built up inside. (Poisson's equation shows that this is possible in a quasineutral plasma, because all lengths in z_2 are much larger than λ_D and a small departure from neutrality produces large cumulative fields). This is a kind of overshooting similar to that appearing in shock waves when some parameter goes to zero.

This overshooting is equivalent to a shifting of the probe potential; this means that the transition region extends beyond space potential and that this is no special value. However, for some large value of χ_p, this potential at the probe finally catches up with $\chi_0(z_2 = 0, \chi_p)$; from there on the formulation of this chapter fails. The overshooting is the essential phenomenon in probe collection in the presence of a strong magnetic field.

Chapter IV

The Probe Characteristic

A. The behavior of the probe characteristic

A1. The floating potential

At the floating potential $\left(\alpha = \frac{eV_f}{kT_e} \equiv \alpha_f\right)$, $I^e = I^i$. As will be seen below, for large negative α, the effect of the magnetic field on both I^e and I^i becomes small. Because μ is very small $|\alpha_f|$ should be large. Therefore an estimate of $O\left(|\alpha_f|\right)$ can be taken from the case $B = 0$; then $|\alpha_f| = O\left(\ln \mu^{-1}\right)$.

The results we shall find can be summarized as follows. There exists a change in the floating potential, $\Delta\alpha_f$ where $\Delta\alpha_f = O(1)$. This change is attributable to two effects appearing in I^e; one is present whenever the probe area differs from the area of its (two-sided) cross-section; the other is a small variation, at $\alpha \approx \alpha_f$, in the current density itself. The change in I^i is negligible. The slope of I at α_f experiences a relatively small decrease because of the magnetic field. A precise determination of both α_f and $\left.\frac{dI}{d\alpha}\right|_{\alpha=\alpha_f}$ is of interest because these are two features of the $I - V_p'$ diagram which are readily accessible from experiments. Equating the right-hand sides of (3.63) and (3.73) we obtain

$$\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{\alpha}e^{-\frac{\beta+1}{2}\psi}}{1+\mathrm{erf}\left\{\frac{\beta}{2}\psi-\alpha\right\}^{1/2}}=-\frac{2^{3/2}\gamma_E^{1/2}(\beta+1)}{(3\pi)^{1/2}}\sigma\frac{\partial\psi}{\partial\zeta} \qquad (4.1)$$

where both sides represent the non-dimensional electron density current to the probe. As $\alpha\to-\infty$, $\psi(\zeta=0,\ \xi_k<1)\to 0$ and both sides of (4.1) must vanish at the same rate. For $-\alpha_f$ large but finite, $\psi(\zeta=0,\ \xi_k<1)\ll 1$ will be satisfied and therefore $\psi\ll 1$ in the whole ζ-ξ_k plane. Then (3.62) can be linearized and I^e found analytically. We expand ψ and from (3.62) - (3.64) is obtained

$$\left(\frac{\partial^2\psi_0}{\partial\zeta^2}+\frac{1}{\xi_k^s}\frac{\partial}{\partial\xi_k}\xi_k^s\frac{\partial\psi_0}{\partial\xi_k}\right)+\left(\frac{1}{\xi_k^s}\frac{\partial}{\partial\xi_k^s}\xi_k^s\left\{\frac{\partial\psi_1}{\partial\xi_k}-\psi_0\frac{\partial\psi_0}{\partial\xi_k}\right\}\right.$$

$$\left.+\frac{\partial^2\psi_1}{\partial\zeta^2}\right)+\dots \qquad (4.2)$$

with the boundary conditions

$$\left.\begin{array}{ll}\lim\limits_{(\xi_k^2+\zeta^2)^{1/2}\to\infty}\psi_j=0 & \\ \left.\dfrac{\partial\psi}{\partial\zeta}\right|_{\zeta=0}=0 & \text{for } |\xi_k|>1\end{array}\right\}\text{all } j \qquad (4.3)$$

An equivalent expansion of (4.1) yields

$$\frac{e^{\alpha}}{2}\left\{1-\frac{\beta+1}{2}\psi_0+\dots\right\}\left\{1+\frac{e^{\alpha}}{2(-\pi\alpha)^{1/2}}\left(1-\frac{\beta}{2}\psi_0+\frac{\beta}{4\alpha}\psi_0+\dots\right)\right\}=-\sigma^*\left\{\frac{\partial\psi_0}{\partial\zeta}+\frac{\partial\psi_1}{\partial\zeta}+\dots\right\} \quad (4.4)$$

where

$$\sigma^*\equiv(\beta+1)\frac{2\,\gamma_E^{1/2}\,\sigma}{3^{1/2}} \quad (4.5)$$

(To expand the error function, we have used the asymptotic expansion

$$\operatorname{erfc}x=1-\operatorname{erf}x\approx\frac{e^{-x^2}}{\pi^{1/2}x}\left\{1-\frac{1}{2x^2}+O(x^{-4})\right\}$$

for large x.)

If we know $\psi_0(\zeta=0,|\xi_k|<1)$ we have in the right-hand side of (4.4) the first two terms in an expansion of the electron current density. To find ψ_0 we have to solve the first term of (4.2):

$$\frac{\partial^2\psi_0}{\partial\zeta^2}+\frac{1}{\xi_k^s}\frac{\partial}{\partial\xi_k}\xi_k^s\frac{\partial\psi_0}{\partial\xi_k}=0 \quad (4.6)$$

with the boundary conditions from (4.3)

$$
\begin{aligned}
&\lim_{(\xi_K^2+\zeta^2)^{1/2}\to\infty} \Psi_0 = 0 \\
&\left.\frac{\partial\Psi_0}{\partial\zeta}\right|_{\zeta=0} = 0 \qquad \text{for } |\xi_K|>1 \\
&\left.\frac{\partial\Psi_0}{\partial\zeta}\right|_{\zeta=0} = \frac{-e^{\alpha}}{2\sigma^*} \quad \text{for } |\xi_K|<1
\end{aligned}
\tag{4.7}
$$

Eq. (4.6) is Laplace's equation. It results immediately that a cylindrical probe (s = 0) is not a well-posed problem, independently of its shape parallel to $\vec{B}$, because the solution of Laplace's equation has a logarithmic divergence at infinity in two dimensions; a probe infinitely long and perpendicular to $\vec{B}$ disturbs the plasma in its entirety. (A cylindrical probe in a collision-dominated plasma poses a similar difficulty). From now on we shall consider only the case s = 1.

For s = 1, (4.6) and (4.7) are analogous to the equations describing a disc with a given electric charbe density at the surface. While the problem of a disc with a given potential is an old one [53], the author is not aware of any published solution to the present problem; moreover the solution has some interest for the computations of Section B. For this reason we digress briefly to exhibit the solution to (4.6) and (4.7).

We can write

$$\psi_0(\xi_k,\zeta)=\int_0^\infty A(q)e^{-\zeta q}J_0(\xi_k q)\,dq \qquad (4.8)$$

for the solution of Laplace's equation in cylindrical coordinates, whenever there is symmetry with respect to the plane $\zeta = 0$ and ψ_0 satisfies certain weak conditions at $\xi_k=0, \xi_k \to \infty$ (see [53]). J_0 is the Bessel function of the first kind of order zero.

We have to integrate

$$\int_0^\infty A(q)\,J_0(\xi_k q)\,dq$$

because we only need $\psi_0(\xi_k,0)$. To find the unknown function $A(q)$ we use boundary conditions from (4.7). They may be expressed

$$\int_0^\infty A(q)\,q\,J_0(\xi_k q)\,dq=\frac{e^\alpha}{2\sigma^*}, \quad \xi_k<1 \qquad (4.9a)$$

$$\int_0^\infty A(q)\,q\,J_0(\xi_k q)\,dq=0, \quad \xi_k>1 \qquad (4.9b)$$

The inversion formulae for Hankel transforms

$$A(q) = \int_0^\infty A(\xi_k)\xi_k J_n(q\xi_k)\, d\xi_k$$

$$A(\xi_k) = \int_0^\infty A(q)\, q\, J_n(\xi_k q)\, dq$$

give immediately from (4.9a) and (4.9b)

$$A(q) = \int_0^1 \frac{e^\alpha}{2\sigma^*}\xi_k J_0(q\xi_k)\, d\xi_k$$

Using Poisson's integral for J_0,

$$J_0(\xi_k q) = \frac{2}{\pi}\int_0^{\xi_k} \frac{\cos qt}{(\xi_k^2 - t^2)^{1/2}}\, dt$$

there results (integrating first over ξ_k in the double integral):

$$A(q) = \frac{e^\alpha}{2\sigma^*}\int_0^1 (1-t^2)^{1/2}\cos qt\, dt$$

Hence we obtain

$$\psi_0(\zeta = 0, \xi_k < 1) = \frac{e^\alpha}{\pi\sigma^*}\int_0^\infty J_0(\xi_k q)\, dq \times \int_0^1 (1-t^2)^{1/2}\cos qt\, dt \qquad (4.10)$$

Interchanging order of integration there results a type of integral whose discontinuous solution is [54]:

$$\int_0^\infty J_n(\xi_k q)\cos tq\, dq = \frac{\cos(n\sin^{-1} t/\xi_k)}{(\xi_k^2 - t^2)^{1/2}}, \quad 0 \leq t < \xi_k$$

$$= \frac{-\xi_k^n \sin\frac{n\pi}{2}}{(t^2 - \xi_k^2)^{1/2}\{t + (t^2 - \xi_k^2)^{1/2}\}^n}, \quad 0 < \xi_k < t$$

Therefore for $n = 0$, the range $t > \xi_k > 0$ gives no contribution to the second integral in (4.10). We get finally:

$$\psi_0(\xi = 0, \xi_k < 1) = \frac{e^{\alpha}}{\pi\sigma^*} E(\xi_k) \qquad (4.11)$$

where $E(\xi_k)$ is the complete elliptic integral of the second kind; $E(\xi_k)$ has a smooth variation from $E(0) = \frac{\pi}{2}$ to $E(1) = 1$.

We can now find ψ_1 because we have an explicit boundary condition for it in (4.4). However the linearization is valid only if ψ_0 is small enough; we therefore neglect ψ_1 in the right-hand side of (4.4).

A simple integration now gives I^e:

$$I^e \approx (-e)\, 2 \int_0^{2\pi} d\varphi \int_0^{\infty} \xi_k \, d\xi_k \left(\frac{2}{\pi}\right)^{1/2} \frac{e^{\alpha}}{2} \qquad (4.12)$$

$$\times \left[1 - \frac{\beta+1}{2}\psi_0 + \frac{e^{\alpha}}{2(-\pi\alpha)^{1/2}}\right] R^2 N_{\infty} \left(\frac{kT_e}{m_e}\right)^{1/2}$$

It is easy to show that $\int_0^1 \xi_k E(\xi_k)\, d\xi_k = \frac{2}{3}$; and $\int_0^1 \xi_k \, d\xi_k = \frac{1}{2}$. Then, finally:

$$I^e \approx (-e)\, 4\pi R^2 N_\infty \left(\frac{KT_e}{m_e}\right)^{1/2} \frac{e^\alpha}{(2\pi)^{1/2}}\, \frac{1}{2}$$

$$\left[1 - \frac{e^\alpha}{2\pi}\left\{\frac{2\sigma^{-1}}{(3\gamma_E)^{1/2}} - \left(\frac{-\pi}{\alpha}\right)^{1/2}\right\}\right] \qquad (4.13)$$

For very large negative α the right-hand side of (4.1) takes the form $\frac{e^\alpha}{(2\pi)^{1/2}}$; therefore the only changes introduced by the magnetic field are: (1) the factor $\frac{1}{2}$ before the square bracket; and (2) the term $\frac{e^\alpha \sigma^{-1}}{\pi(3\gamma_E)^{1/2}}$ within it. The last term, $\frac{e^\alpha}{2(-\pi\alpha)^{1/2}}$, is very small and can be neglected.

The effective area of a spherical probe in a strong magnetic field is that of a two-sided disc, $2\pi R^2$; the factor $\frac{1}{2}$ shifts α_f to more positive values by an amount equal to $\ln 2$ because I^i depends very little on α for large, negative α.

For large enough, negative values of α, I^i is sometimes taken, for $B = 0$, as

$$I^i \approx (Z_i e)\, 4\pi R^2 N_\infty \left(\frac{KT_e}{m_i}\right)^{1/2} \frac{1}{2}$$

as given in [5]; an examination of Laframboise's numerical results for $T_i = T_e$ (see [10], Fig. 20) shows that the factor $\frac{1}{2}$ should be changed into $(2\pi)^{-1/2}$ times a factor between $\frac{3}{2}$ and 2 (for λ_D/R between $\frac{1}{20}$ and $\frac{1}{100}$). For C_s then $\alpha_f \approx -5$, using (4.13) arising from the presence of the magnetic field can be as large as 25%. The total shift is $\Delta\alpha_f = O(1)$, and is therefore important.

To find α_f exactly, I^i has to be obtained. We shall consider now the case $\lambda \gg R$; for $B = 0$ this is the collisionless theory for which Laframboise's computations are available [10]. This is the dilute plasma case of Section I-B and Figure 5. For $\lambda < R$ this theory is certainly not correct.

The basic result obtained from a study of Figs. 20, 22 and 27b (and also Figs. 25, 26, 27a, 39 and 41) of [10] are the weak dependence of I^i, for $\frac{\lambda_D}{R} \ll 1$ and $\alpha \lessapprox \alpha_f$, (conditions satisfied in the present case) upon: a) the value of α; b) the distribution function of the ions at infinity and c) the geometry of the probe. This means that for changes of order unity in these conditions, $\frac{\Delta I^i}{I^i} \ll 1$. If these conditions themselves experience a small change, $\frac{\Delta I_i}{I^i}$ is a second-order quantity.

For $\alpha \approx \alpha_f, \chi_o(z_2 = 0) (= \chi_o(z_1 \to \infty))$ is small and I^e is near its value for the same α and $B = 0$ (the factor $\frac{1}{2}$ excluded). The small perturbations at $z_1 \to \infty$ on a) the potential; b) f^i and c) the spherical symmetry of the problem produce a higher order correction on I^i. Therefore, to find α_f (4.13) can be used for I^e and Laframboise's results can be used for I^i.

The slope of the $I - V_p$ diagram at the floating potential can be found from (4.13), because

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=\alpha_f} \approx \left.\frac{dI^e}{d\alpha}\right|_{\alpha=\alpha_f}$$

which results from the insensibility of I^i to a change in α. It can be observed that while the factor $\frac{1}{2}$ appears also in $\frac{dI^e}{d\alpha}$, it does not produce a change in $\left.\frac{dI^e}{d\alpha}\right|_{\alpha=\alpha_f}$ because of the shift of α_f itself. The correction to $\left.\frac{dI}{d\alpha}\right|_{\alpha=\alpha_f}$ is basically due to the bracket in (4.13) differing from unity and therefore is less important than the change in α_f.

A2. The ion saturation current

For $\alpha < \alpha_f$ or slightly larger ($-\infty < \alpha \lessapprox 3.5$), the argument given above is valid. Thus we have both components of I over this range of α. In particular for $\alpha \to -\infty$, $I^e \to 0$ and $I^i_s = I^i_s(B = 0)$ as obtained from Laframboise's computations.

A3. The transition region and the space potential

For $\alpha \gtrapprox - 3.5$, I^i decreases rapidly; moreover the detailed form of the decrease depends sensibly on the geometry of the problem and other conditions. Therefore the results from [10] can be used no longer. But for these more positive values of α, I^e becomes much larger than I^i.

Therefore $I \approx I^e$ and the accuracy of this approximation improves as α grows.

To find I^e we have to solve now the non-linear equations (3.62) and (3.64) where the second condition in (3.64b) is given by (4.1). The results of the numerical computations are presented and discussed in section B.

A4. The electron saturation current

When α is not of order unity but large enough, the overshooting of the electric potential disappears, as seen in Figure 8. This can be shown roughly by observing from (4.1) that

$$e^{\alpha} \sim -\frac{\partial\psi}{\partial s}\, e^{\frac{\beta+1}{2}\psi}\left[1+\operatorname{erf}\left\{\frac{\beta}{2}\psi-\alpha\right\}^{1/2}\right] \quad (4.14)$$

so that as α grows ψ has to grow. Then in the left-hand side of (4.1) $e^{\alpha} e^{-\frac{\beta+1}{2}\psi}$ should grow and thus

$$\frac{\partial}{\partial\alpha}\left(\alpha - \frac{\beta+1}{2}\psi\right) > 0$$

or

$$\frac{\partial\psi}{\partial\alpha} < \frac{2}{\beta+1}$$

and since $\lambda_0 \equiv \frac{\beta\psi}{2}$

$$\frac{\partial\lambda_0}{\partial\alpha} < \frac{\beta}{\beta+1} < 1$$

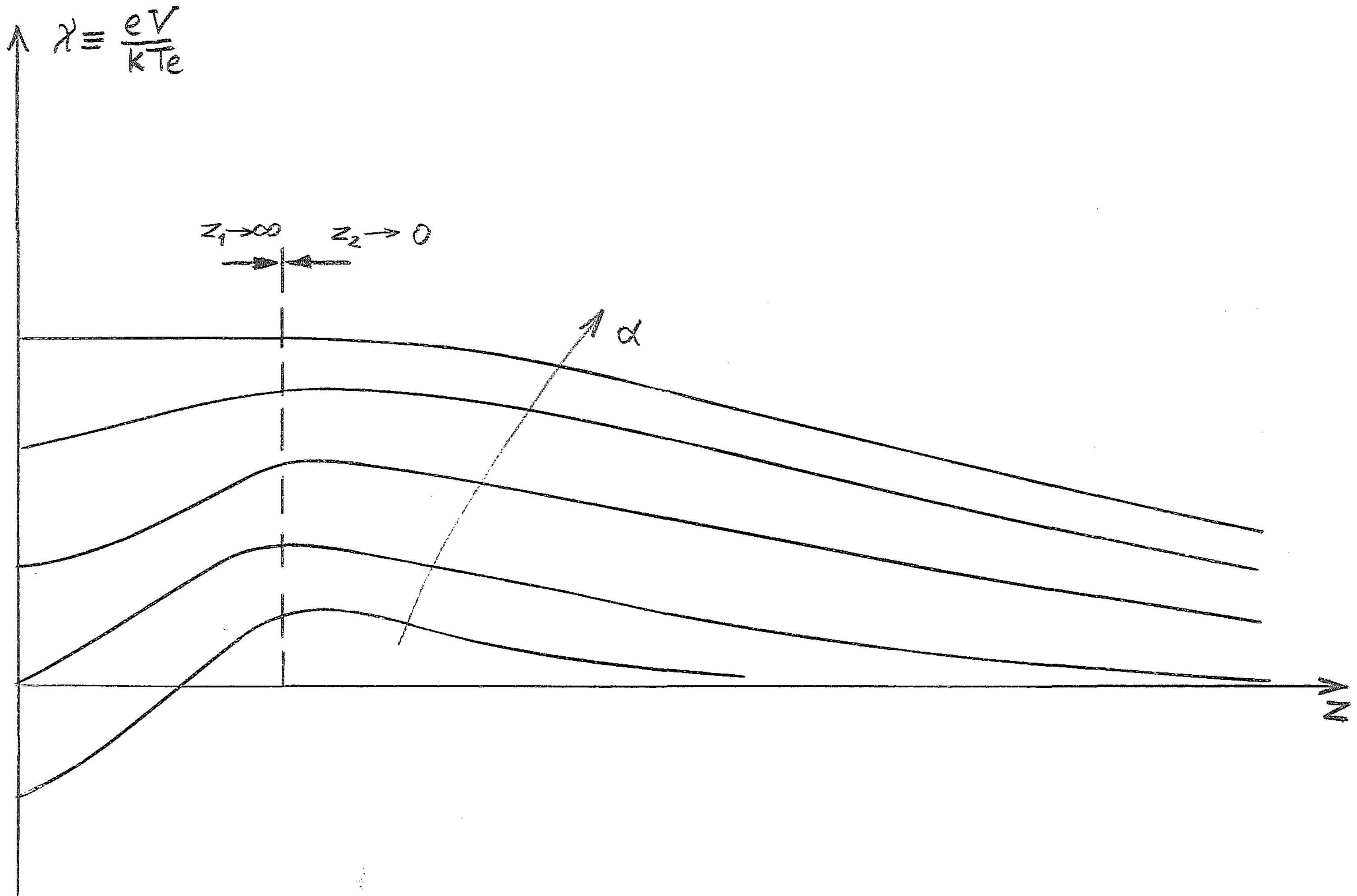

Fig. 8. The overshooting of the potential.

It has been assumed in (4.14) that $\left.\frac{\partial\psi}{\partial\zeta}\right|_{\zeta=0}$ grows monotonically with $\psi(\zeta = 0)$. This is not necessarily true for all $\xi_K < 1$; however, on the average over the probe region $\xi_K < 1$, $\left.\frac{\partial\psi}{\partial\zeta}\right|_{\zeta=0}$ should increase as $\psi(\zeta = 0)$ increases. Thus the above argument is only qualitative; moreover the variation of the error function becomes important for $\frac{\beta}{2}\psi \approx \alpha$. Since this equality is reached for different ξ_K's successively, this variation of $\mathrm{erf}\{\frac{\beta}{2}\psi - \alpha\}^{1/2}$ is of less importance if the argument is made on the average. That α grows faster than $\chi_0 \equiv \frac{\beta}{2}\psi$ has been observed in the present computations.

Therefore for α larger than a certain value the formulation of Section III-F fails. Nevertheless, it is possible to show that this formulation, as represented by equations (3.62) and (3.64), contains a saturation of the electron current. If σ is small enough, the asymptotic limit of I^e is clearly defined before the overshooting disappears.

The reason for this saturation is the strong non-linear character of Eq. (3.62):

$$\frac{\partial^2\psi}{\partial\zeta^2} + e^{-\psi}\left[\frac{\partial^2\psi}{\partial\xi_K^2} + \frac{1}{\xi_K}\frac{\partial\psi}{\partial\xi_K} - \left(\frac{\partial\psi}{\partial\xi_K}\right)^2\right] = 0 \qquad (3.62)$$

As ψ increases the second term becomes negligible compared to the first because of the factor $e^{-\psi}$. Thus for ψ very large

$$\frac{\partial^2\psi}{\partial\zeta^2} \approx 0, \quad \text{or} \quad \frac{\partial\psi}{\partial\zeta} = \text{constant} \qquad (4.15)$$

Now in the limit of $\psi(\zeta=0)\rightarrow\infty$, $\left.\frac{\partial\psi}{\partial\zeta}\right|_{\zeta=0}$ cannot go to infinity but should approach a finite asymptotic value.

In effect this slope would be constant, from (4.15) until $\psi = O(1)$. If the solution of ψ were curve 1 in Figure 9, ψ would approach values of order unity with a very large negative slope. Then before the second term in (3.62) could produce a sensible curvature, ψ would overshoot to large, negative values, following curve 1'' instead of 1'.

From this argument there results also that large values of ψ, or χ_0, are obtained only inside the "shadow" as we already knew. If $\psi(\zeta = 0, \xi_k > 1)$ were to be very large it would remain so because of (4.15) and the first equation of (3.64b). This result is discussed in Appendix F in conjunction with the analysis of the z_0, z_1-layers.

For α larger than the value at which the overshooting disappears, the electrons would be attracted in

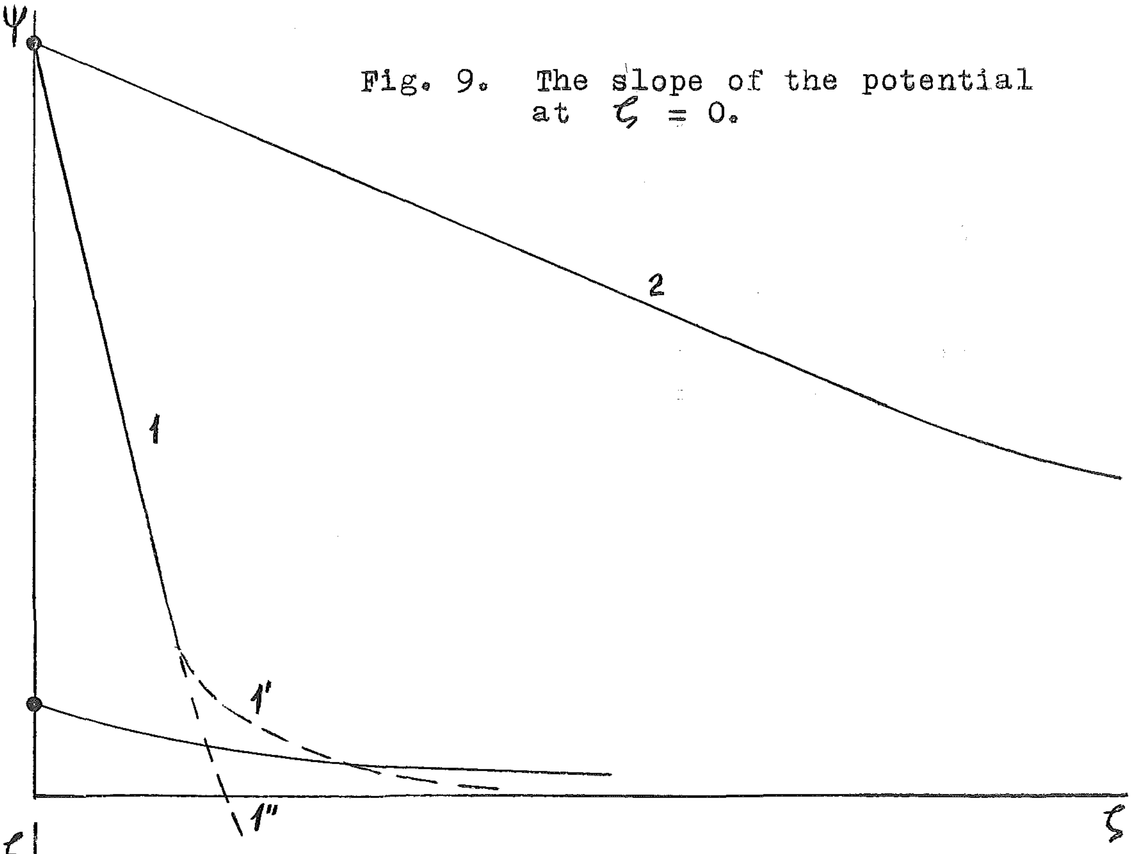

Fig. 9. The slope of the potential at $\zeta = 0$.

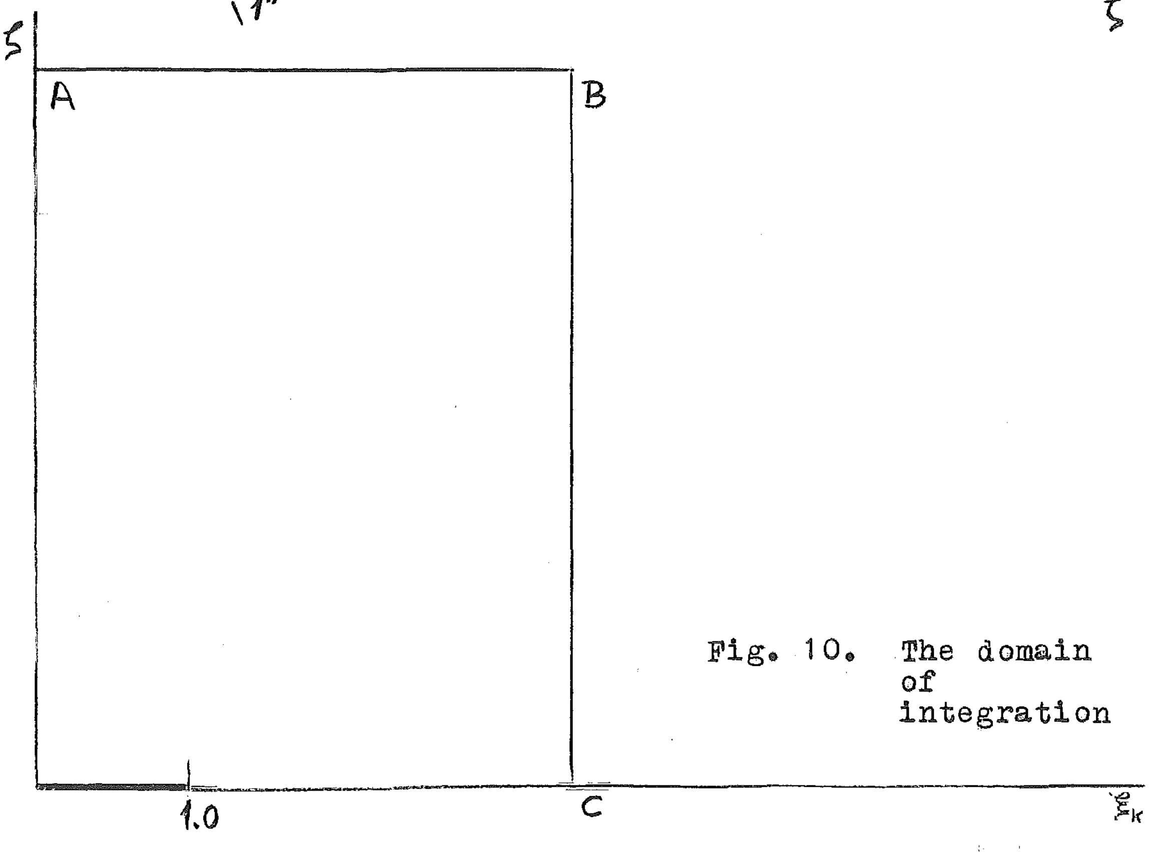

Fig. 10. The domain of integration

the z_0, z_1-layers. If we retain a Maxwellian form for the electrons at $z_1 \to \infty$, we would find

$$n_0^e(z_0,z_1) = C e^{\chi_0(z_0,z_1)} e^{-\chi_0(z_2=0)}$$

$$\times\left[1-\operatorname{erf}\{\chi_0-\chi_0(z_2=0)\}^{1/2}\right]$$

so that

$$\lim_{z_1\to\infty} n_0^e \equiv \frac{C}{2} = \lim_{z_2\to 0} n_0^e = e^{-\frac{\chi_0(z_2=0)}{\beta}}$$

The current then would be

$$n_0^e u_z^{e0} = -\frac{C}{(2\pi)^{1/2}} = -\frac{2e^{-\frac{\chi_0(z_2=0)}{\beta}}}{(2\pi)^{1/2}} \qquad (4.16)$$

This is also the limiting form of (3.73) when $\alpha = \chi_0(z_2 = 0)$. (4.16) does not depend on α; however, since $\chi_0(z_2 = 0) = \alpha$ is satisfied for $\xi_k < 1$, successively, there is a smooth approach to a constant value of the current. This result is unreliable because of the use of a Maxwellian distribution function is unjustified.

B. Results of the computations

The equation

$$\frac{\partial^2\psi}{\partial\zeta^2} + e^{-\psi}\left[\frac{\partial^2\psi}{\partial\xi_k^2} + \frac{1}{\xi_k}\frac{\partial\psi}{\partial\xi_k} - \left(\frac{\partial\psi}{\partial\xi_k}\right)^2\right] = 0$$

has been solved numerically using the University of Colorado CDC 3600 computer. The boundary conditions imposed are

$$\lim_{(\xi_k^2+\zeta^2)^{1/2}\to\infty} \psi = 0$$

at $\zeta = 0$ $$\begin{cases} \dfrac{\partial \psi}{\partial \zeta} = 0 & \text{for } \xi_k > 1 \\ \dfrac{e^{\alpha} e^{-\frac{\beta+1}{2}\psi}}{1+\operatorname{erf}\left\{\frac{\beta\psi}{2}-\alpha\right\}^{1/2}} = -\dfrac{2(\beta+1)\gamma_E^{1/2}}{3^{1/2}} \sigma \dfrac{\partial \psi}{\partial \zeta} & \text{for } \xi_k < 1 \end{cases}$$

$$\frac{\partial \psi}{\partial \xi_k} = 0 \qquad \text{at} \quad \xi_k = 0$$

The condition at infinity is however difficult to take into account. Thus the integration has been made over a finite domain. A coordinate asymptotic expansion for ψ was obtained for large $(\xi_k^2 + \zeta^2)^{1/2}$. The known behavior of ψ at infinity then allowed the use of a finite domain (see Fig. 10).

For $(\xi_k^2 + \zeta^2)^{1/2}$ large enough, ψ will be very small. It is possible to perform an expansion of the form:

$$\psi = \psi_0 + \psi_1 + \dots$$

where

$$\lim_{(\xi_k^2+\zeta^2)^{1/2}\to\infty} \frac{\psi_j}{\psi_{j-1}} = 0$$

The result, given below, is then used as the boundary condition as $(\xi_k^2+\zeta^2)^{1/2}\to\infty$ for the numerical calculations in the finite domain.

The equations for the ψ_j's are of the form

$$\frac{\partial^2\psi_j}{\partial\zeta^2}+\frac{\partial^2\psi_j}{\partial\xi_k^2}+\frac{s}{\xi_k}\frac{\partial\psi_j}{\partial\xi_k} = \Theta_j(\psi_0,\cdots,\psi_{j-1}) \qquad (4.17)$$

For $j = 0$, $_0$ 0. Since Laplace's equation has a logarithmic divergence at infinity in the case $s = 0$, we shall consider s to be 1.

The complete solution of the system (4.17) is the general solution to the homogeneous system plus particular solutions to the inhomogeneous equations. The first can be expressed, using spherical coordinates, as

$$\sum_\ell P_\ell(\cos\theta)\left[\frac{a_\ell}{\rho^{\ell+1}}+b_\ell\rho^\ell\right]$$

where

$$\tan\theta = \frac{\xi_k}{\zeta}\ , \quad (\xi_k^2+\zeta^2)^{1/2} = \rho$$

and P_ℓ are the Legendre polynominals.

Because $\psi\to 0$ as $\rho\to\infty$, $b_\ell \equiv 0$ for all ℓ. Also ψ should have $\zeta = 0$ as a plane of symmetry. Thus $a_\ell = 0$, for ℓ odd. The above expression becomes

$$P_0 \frac{a_0}{\rho} + \frac{P_2 a_2}{\rho^3} + \dots$$

To find the solution to

$$\frac{\partial^2 \psi_1}{\partial \zeta^2} + \frac{\partial^2 \psi_1}{\partial \xi_k^2} + \frac{1}{\xi_k} \frac{\partial \psi_1}{\partial \xi_k} = \Theta_1 (\psi_0)$$

we change the Laplace operator to spherical coordinates.
Θ_1 is given by

$$\Theta_1 (\psi_0) = \frac{1}{\xi_k} \frac{\partial}{\partial \xi_k} \xi_k \psi_0 \frac{\partial \psi_0}{\partial \xi_k}$$

Using

$$\psi_0 = \frac{a_0}{\rho} = \frac{a_0}{(\xi_k^2 + \zeta^2)^{1/2}}$$

Θ_1 is expanded in Legendre polynominals and it is immediately obtained for ψ_1

$$\psi_1 = \frac{a_0^2 \cos^2 \theta}{\rho^2}$$

Thus for large ρ

$$\psi = \frac{a_0}{\rho} + \frac{a_0^2 \cos^2 \theta}{\rho^2} + O\left(\frac{1}{\rho^3}\right)$$

The symmetry with respect to $\zeta = 0$ has allowed us to obtain two terms of the expansion with only one constant; the terms $O(\rho^{-3})$ would involve a new constant a_2.

Now in Figure 10 a boundary condition is available along AB choosing this boundary far enough; eliminating a_0 between ψ and $\frac{\partial\psi}{\partial\zeta}$ we have

$$\zeta\frac{\partial\psi}{\partial\zeta}+2\psi\frac{1-(\xi_k/\zeta)^2}{1+(\xi_k/\zeta)^2}=\left[\left(\frac{\xi_k}{\zeta}\right)^2-\frac{1}{2}\right]\left[1-\left\{1+\frac{4\psi}{1+(\xi_k/\zeta)^2}\right\}^{1/2}\right]$$

Similarly along BC

$$\xi_k\frac{\partial\psi}{\partial\xi_k}+\frac{4\psi}{1+(\zeta/\xi_k)^2}=-\frac{3}{2}\left(\frac{\xi_k}{\zeta}\right)^2\left[1-\left\{1+\frac{4\psi}{1+(\xi_k/\zeta)^2}\right\}^{1/2}\right]$$

At point C this condition becomes

$$\xi_k\frac{\partial\psi}{\partial\xi_k}+\psi=0$$

The elliptic differential equation (3.62) was then changed into a finite difference equation. This was solved by row relaxation. In the iterative procedure, the system of equations was linearized using $\psi^{(m-1}$ for the terms non-linear in ψ, for the m^{th} iteration.

An array of 50 times 50 meshes was chosen. In order to have a sensible estimate of ψ^0 to initiate the iteration, a large negative value of α was chosen first so that ψ was small and the solution to the linearized equation was available (see Section IVA). Then, for fixed values of α, and $(\beta+1)\,\gamma_E(z_1)\sigma$, a scaling of the solution to the case $\alpha=\alpha_\ell$ was chosen as initialization for the case $\alpha_{\ell+1}$, where $\alpha_{\ell+1}-\alpha_\ell>0$. The scaling was made by writing

$$\psi^{0}(\xi_k,\zeta,\alpha_{\ell+1}) = \psi^{F}(\xi_k,\zeta,\alpha_\ell)\left[1+\frac{\alpha_{\ell+1}-\alpha_\ell}{\frac{\beta+1}{2}\,\overline{\psi^{F}}}\right]$$

where ψ^F is the result of the final iteration and $\overline{\psi}^F$ is an average of $\psi^F(\zeta = 0, \xi_k < 1)$. When $\alpha_{\ell+1} - \alpha_\ell = 1$ the number of iterations required was around 10, but for $\alpha_{\ell+1} - \alpha_\ell \gtrsim 2$ it increased up to 10^2. Approximately 15 iterations were performed per minute.

An accuracy of 1% was chosen for the derivative $\frac{\partial\psi}{\partial\zeta}$ at ($\zeta = 0$, $\xi_k < 1$). This derivative is much more sensitive than ψ itself so that this accuracy corresponded to errors of order of a few thousandths in ψ. Although it was not used, a convergence criterion was available, of a type sometimes found for elliptic equations; it connects the behaviors of ψ at opposed regions of the contour. If (3.62) is integrated over ζ between zero and infinity we have

$$-\left.\frac{\partial\psi}{\partial\zeta}\right|_{\zeta=0} = \int_0^\infty d\zeta\,\frac{1}{\xi_k}\frac{\partial}{\partial\xi_k}\xi_k\frac{\partial}{\partial\xi_k}e^{-\psi}$$

Integrating now over $\xi_k d\xi_k$ between $\xi_k = 0$ and a large value of ξ_k, ξ_k^1, the right-hand side is zero for $\xi_k > 1$. Therefore

$$I^e \sim \xi_k d\xi_k \frac{\partial \psi}{\partial \zeta}\bigg|_{\zeta=0} = \int_0^\infty d\zeta \int_0^{\xi_k^1} \xi_k \, d\xi_k \frac{1}{\xi_k} \frac{\partial}{\partial \xi_k} \xi_k \frac{\partial}{\partial \xi_k} e^{-\psi}$$

$$= \int_0^\infty d\zeta \; \xi_k^1 \frac{\partial}{\partial \xi_k} e^{\psi}\bigg|_{\xi_k = \xi_k^1}$$

We now write $\psi = \frac{a_0}{\rho}$, integrate and let $\xi_k^1 \to \infty$; the result is

$$I^e \sim \int_0^1 \xi_k d\xi_k \frac{\partial \psi}{\partial \zeta}\bigg|_{\zeta=0} = -a_0 \qquad (4.18)$$

This relation between the behavior of ψ at $(\zeta = 0, \xi_k < 1)$ and $(\xi_k^2 + \zeta^2)^{1/2} \to \infty$ can be used for overrelaxation in the iterative process.

The behavior of ψ at $\zeta = 0$, $\xi_k = 1$ appears to be regular. In Section IV-A1 we showed it for large negative α, such that linearization was possible; this is contrary to the case of a disc at constant potential (see [53]) where a singularity exists at $\zeta = 0, \xi_k = 1$ (however, it is integrable and has a very localized effect; the theoretical calculation of the electric capacity of a disc agrees very well with the experimental value). When α became more positive no anomaly was observed around this critical point.

Eq. (4.18) can be understood observing that (3.62) can be written

$$\frac{\partial^2\psi}{\partial\zeta^2}+\frac{1}{\xi_k}\frac{\partial}{\partial\xi_k}\xi_k\frac{\partial\psi}{\partial\xi_k}=\Theta(\psi) \qquad (4.19)$$

$\Theta(\psi)$ represents thus a "charge density" in a Poisson-type equation; integrating over the whole space it is seen that

$$\int_0^\infty d\zeta\int_0^\infty \xi_k\, d\xi_k\, \Theta = 0$$

Thus the total "charge" is that of the "disc" at $\zeta = 0$, $\xi_k < 1$ and is given by integrating the electric field $\left(\sim\frac{\partial\psi}{\partial\zeta}\right)$ over this area. Because a_0 represents in a solution to Laplace's equation the total charge, (4.18) follows.

In Fig. 11 the field along the z-axis is represented in normalized units ψ and ζ. It can be observed that the slope $\frac{\partial\psi}{\partial\zeta}$, which decreases immediately for small $\psi(\zeta=0)$, remains constant over some distance for larger $\psi(\zeta=0)$, as observed in Section IV-A4 in connection with the saturation effect. The saturation however, does not appear yet for these small values of α. As α increases, the overshooting $\left[\frac{\beta}{2}\psi(\xi=0)-\alpha\right]$ decreases as observed in Figure 11.

Figure 12 represents the field along the ξ-axis for $\sigma^{-1} = 104$, $z_1 = 1$, $\beta = 1$ and several values of α. The field has a behavior completely different than that along the z-axis; the slope $\frac{\partial \Psi}{\partial \xi_k}$ is not monotonic and is very large around $\xi_k = 1$, when the current collected is large. It is observed that as $\Psi(\zeta = 0, \xi_k < 1)$ increases, $\Psi(\zeta = 0, \xi_k > 1)$ remains small as was observed in Section IV-A4. The absence of a build up of the field for $\xi_k > 1$ is also discussed in Appendix F.

In Figure 13 a panoramic representation is made of Ψ for the above values of σ^{-1}, z_i, β and for $\alpha = 0.32$.

It has been observed in the numerical results that as $\Psi(\zeta = 0)$ increases the ratio $\frac{\Psi(\zeta=0,\xi_k=0)}{\Psi(\zeta=0,\xi_k=1)}$, which for $\Psi \to 0$ goes to $\frac{E(0)}{E(1)} = \frac{\pi}{2} \approx 1.57$ (see Section IV-A1), decreases. This can be seen also in Figure 12. As for the ratio $\left.\frac{\partial \Psi}{\partial \zeta}\right|_{\substack{\zeta=0\\ \xi_k=1}} \Big/ \left.\frac{\partial \Psi}{\partial \zeta}\right|_{\substack{\zeta=0\\ \xi_k=0}}$ which goes to 1 as $\Psi \to 0$, increases with Ψ, but it tends to taper off because of the non-linear effect in (3.62); thus $\frac{\partial \Psi}{\partial \zeta}$ is not singular at $(\zeta = 0, \xi_k = 1)$ as in the electrostatic problem of a disc at given potential.

In Figure 14 the normalized electron current $I^e \left[R^2 N_\infty \left(\frac{K T_e}{m_e} \right)^{1/2} \right]^{-1}$ is represented versus α for $\beta = 1$, and several values of $\gamma_E(z_i)\sigma$; in the figure the curves have been labelled assuming $z_i = 1$. As it is seen the results seem to be valid for smaller values of B up to B = 0. For $\alpha \to -\infty$ all the curves collapse together, as observed in Section IV-A1.

Also for $-5 < \alpha < -3$, where floating potential is expected, a significant change on I^e, and therefore on α_f, is observed for large B. The curve in the transition region has not an exponential form, as it has been often assumed (see Chen's review in [25]). For $\alpha \approx 2$ an increase in the slope of I^e appears. This is certainly a spurious result; it appears when the argument of the error function becomes small (small overshooting); then, as commented in Section III-F, the boundary condition (4.1) is not valid. It can be seen there that the denominator in the right-hand side of (4.1) decreases from 2 to 1, increasing, although indirectly, the current. This hides any beginning saturation for these small values of α. For σ^{-1} large enough it would be possible to observe clearly the saturation; unfortunately ψ approaches unity on the boundary AB on Figure 10, for increasing α, and the boundary condition there breaks down. A problem of storage in the computer was found; the possible extension from 50 x 50 to 80 x 80 meshes does not help sensibly because, as seen in Figure 11, ψ decreases very slowly at these distances. The study of larger α's, including the range past the overshooting using the formulation suggested in Section III-F, is intended in future work.

Figure 15 shows the dependence on β; it is significant so as to provide information on the temperature ratio.

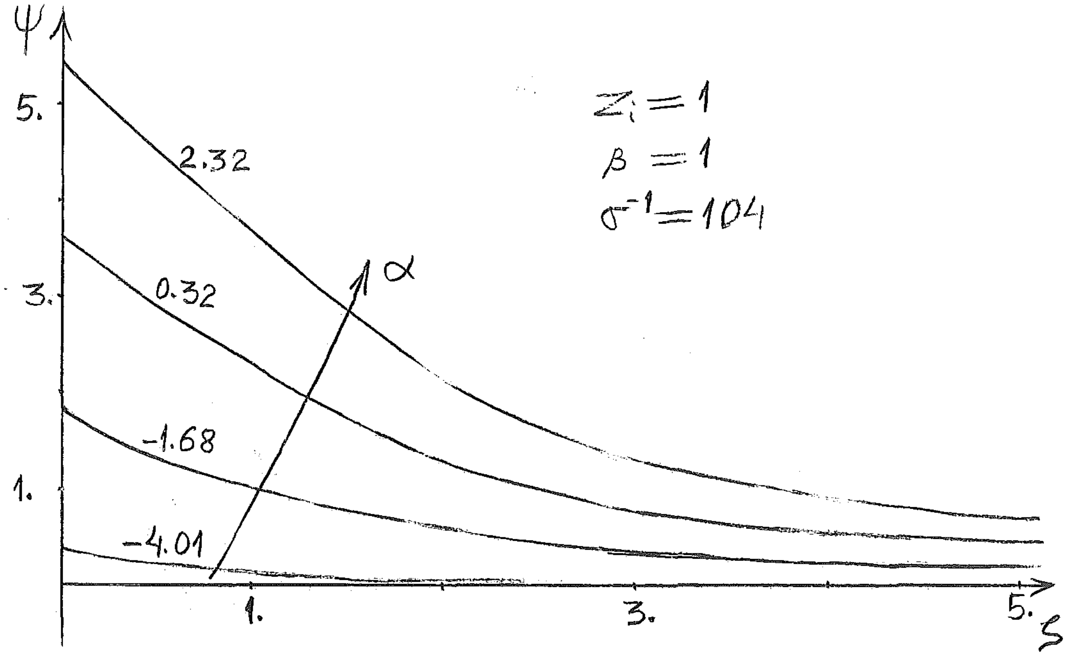

Fig. 11. The decay of the potential along the z-axis.

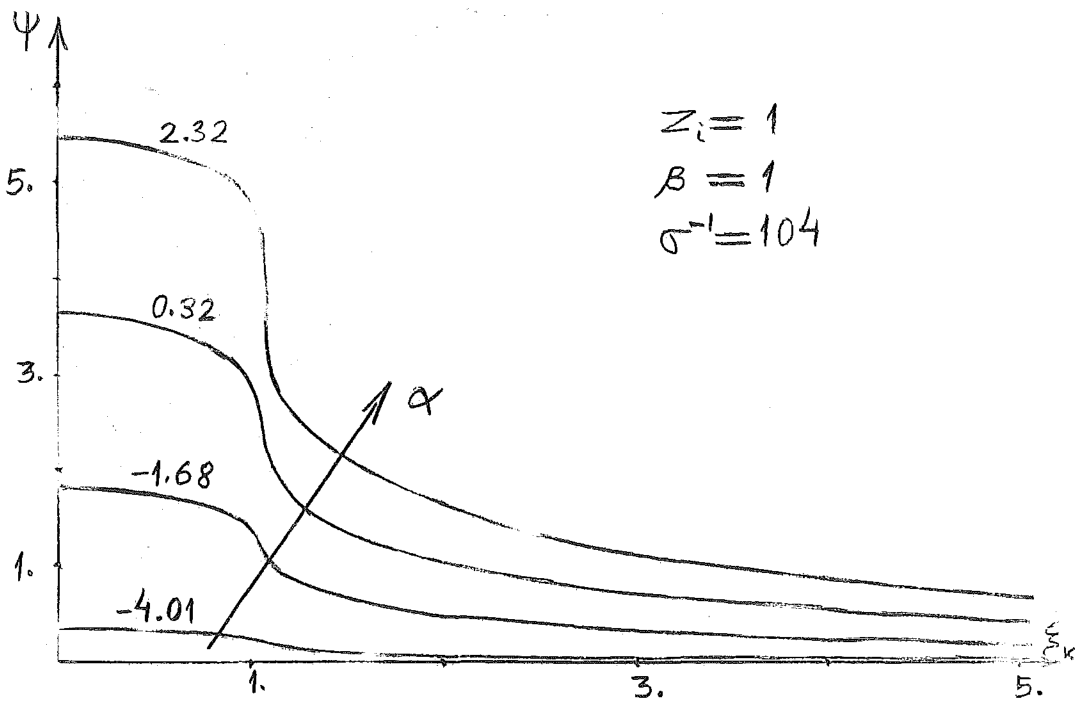

Fig. 12. The decay of the potential along the -axis.

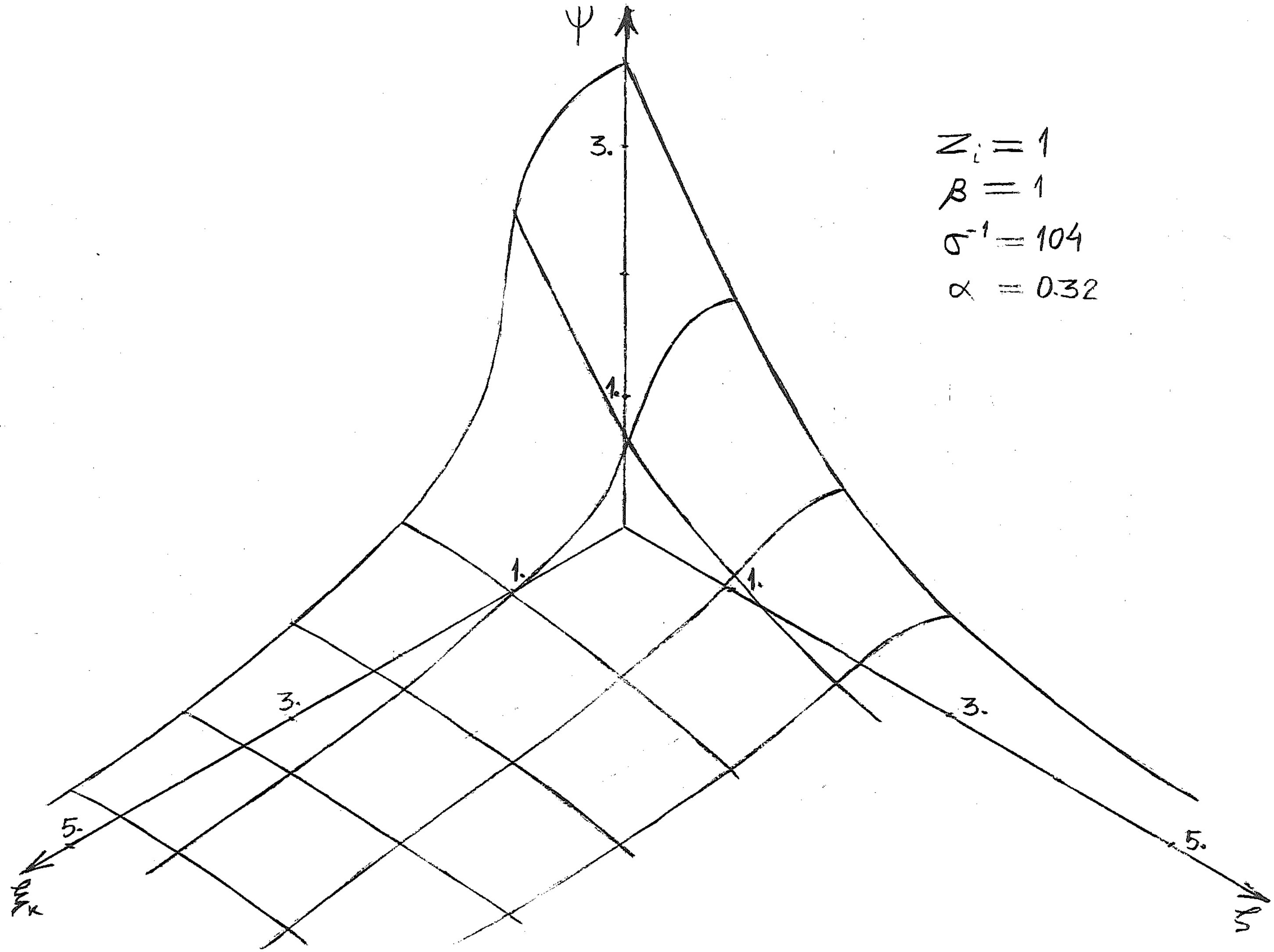

Fig. 13. The potential field.

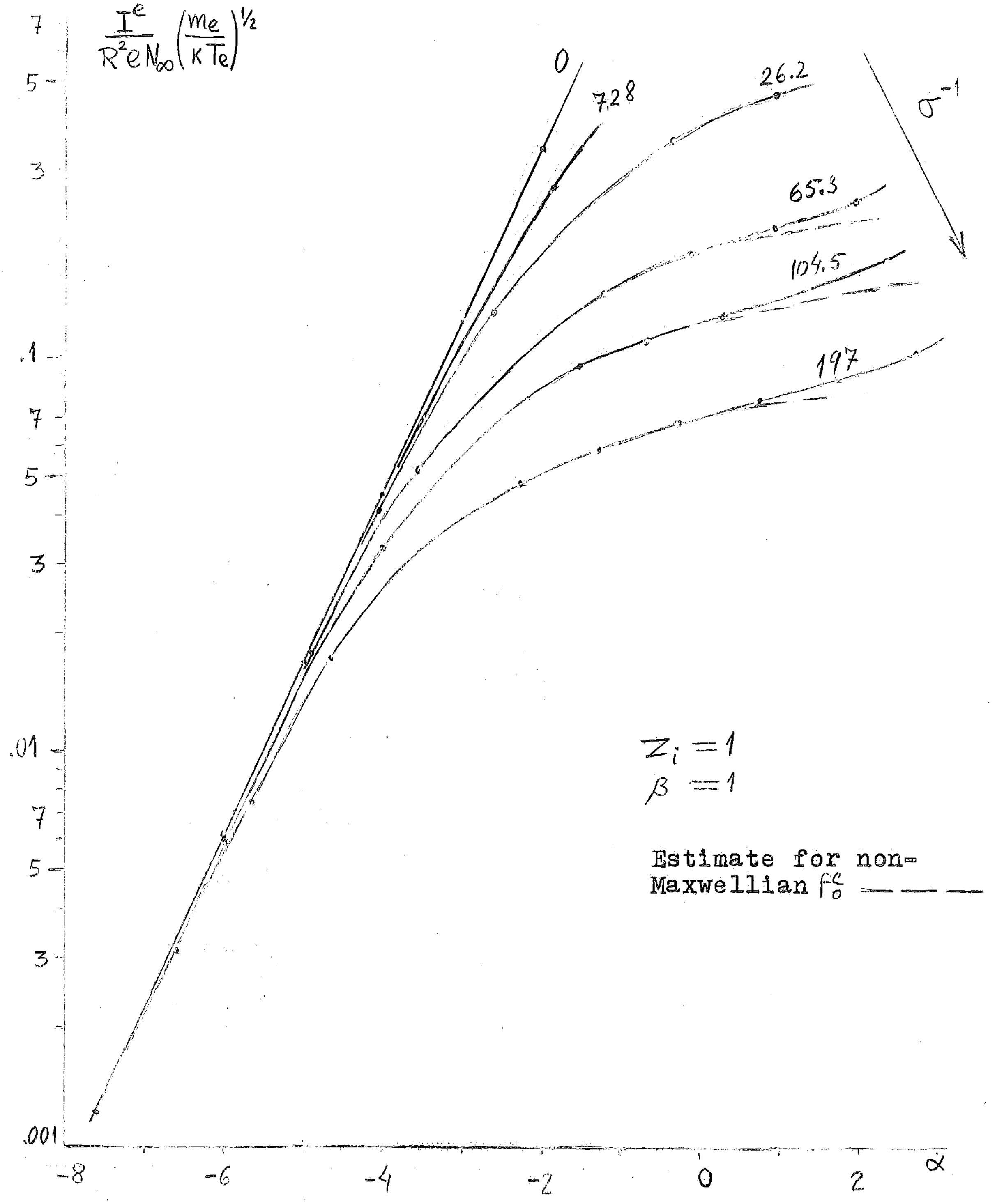

Fig. 14. The electron current as a function of σ^{-1}.

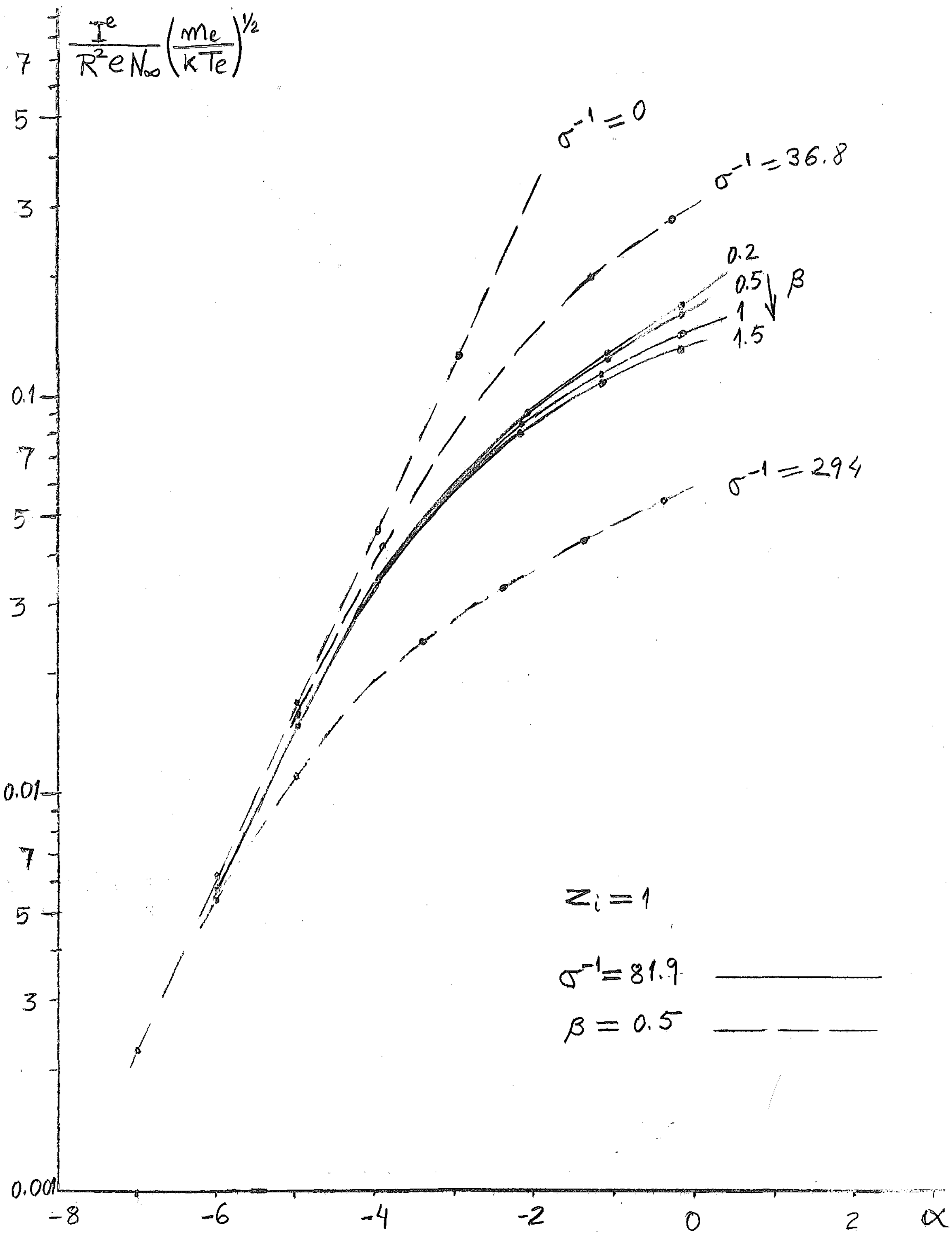

Fig. 15. The electron current as a function of β .

Chapter V

Discussion

A. Extensions of the theory

A number of new effects can be immediately included in the present theory. Some others require a modification which, at present, is not straight forward.

A first, simple correction stems from the consideration of the work function of the probe. If its surface is in a pure condition, this is done by simply shifting the origin of potentials; moreover, it affects only the determination of the space potential. Layers of impurities can affect the $I - V_p$ diagram, unless kT_e is large. Often it is convenient to heat the probe by drawing large saturation current. Non-uniform work functions were considered by Medicus [55].

Second, reflection is easily taken into account because the electrons have a well-defined motion toward the probe; here the longitudinal shape of the probe may have some effect.

Third, recombination can be included, by adding a term $\tilde{\sigma} e^{-\psi}$ to the continuity equation (3.62) in the z_2-layer, because there $n_0^e = n_0^i = e^{-\psi/2}$; $\tilde{\sigma}$ is a non-dimensional recombination coefficient. In the interior z-layers, gradients are strong and moreover, the densities are small so that recombination in z_0 and z_1 is not important.

The extension of the theory to weakly-ionized gases introduces a simplification. Neutrals are not affected by the fields and because the charged-particle concentrations are small the neutrals are not affected at all and transport coefficients are constant. Thus, both weak and strong variations in (3.58) disappear and (3.58) becomes Laplace's equation which is solvable analytically. However, the z_1-layer has no simplification as in our case; moreover, the neutrals only force the electrons to be isotropic. A new ordering of length scales appears necessary.

Anomalous diffusion can possibly be dealt with if a rational theory for the transport coefficients in such conditions is available. Nevertheless, the interior z-layers do not seem simple to analyze in unstable situations.

The extension of the range of magnetic field strengths considered in the present theory may be readily accomplished for certain ranges of the non-dimensional parameters. As we saw in Chapter III, the restriction $\mu \ll \sigma$ (i.e., $R \ll \ell_1$) is not necessary for the determination of I^e. Neither is $\mu < \sigma$, so that the case $\mu > \sigma$ is included; it is only required that σ not be much less than μ. To obtain I^i, the condition $\mu < \sigma$ is necessary and seems to be sufficient. The case $\sigma \ll \mu$, where B is enormously large does not appear to present any difficulty in a treatment like the present one. In the limit $B \to \infty$, our formulation for I^e has a proper behavior, giving for a fixed α, a zero limiting value of I^e; (see equations (3.62), (3.64) and (4.1) with $\sigma \to 0$). However, μ should go to zero at the same time.

The extension to low values of B seems correct, but only for a restricted range of α. For I^i, the discussion of Chapter IV remains valid. For I^e it is only necessary to consider equation (4.1) as $\sigma \to \infty$ ($B \to 0$). It is immediate that for a fixed α, $\psi(\zeta = 0, \xi_k < 1) \to 0$ as $\sigma \to \infty$ and this in "proper" way so that

Now (3.62) is linear for small ψ, and $\psi \sim \sigma^{-1}$ for large σ. This implies that the overshooting goes to zero as $B \rightarrow 0$. Moreover, the z_2-layer itself shrinks to zero: $L_2^z = \frac{\lambda R}{\ell_e} = \lambda \sigma^{-1}$ and thus as $\sigma \rightarrow \infty$, L_2^z is first of order λ and then much smaller. Thus the perturbation vanishes in distances of the scale of λ when $B \rightarrow 0$ (R should be smaller than λ).

Because the overshooting decreases, the upper limit of α available shifts to the left, too; at $B = 0$ this limit is $\alpha = 0$ (see the argument of the error function in (4.1)). However, for $\alpha < 0$ and $B = 0$, $I^e = \frac{e^{\alpha}}{(2\pi)^{1/2}}$ in non-dimensional units except for α very near zero. In (4.1) with $\psi = 0$, the value of I^e is different for small $-\alpha$; for $\alpha \lessgtr -1$ the difference is less than 8% and for $\alpha \lessgtr -2$ less than 1%. For small but non-zero B the usable range of α extends to the right. The reason for this difficulty is that, for small B, the transverse motion of the electrons across B near the probe is important.

In brief, the present theory seems correct from $B \approx 0$ (in the range $-\infty < \alpha \lessapprox 2$) to $B \rightarrow \infty$ (in the range $-\infty < \alpha < \infty$) and if $\ell_i \geqslant O(R)$.

B. Conclusion

The present study is a consistent asymptotic analysis in the limit when some small parameters, which appear naturally in the problem, approach zero. The whole $I - V_p$ diagram is considered, with varying degrees of restrictive assumptions in the several parts of the characteristic.

The dependence on the non-dimensional parameters evolves naturally: the parameters $\alpha = \frac{eV_p}{kT_e}$, $\beta = \frac{T_i}{T_e z_i}$, z_i and $\mu = \frac{\ell_e}{\ell_i}$ intervenes when both I^e and I^i are important (in particular in the determination of α_f and $\left.\frac{dI}{d\alpha}\right|_{\alpha=\alpha_f}$). Although $\varepsilon = \frac{\lambda_D}{\lambda}$ and $\gamma^{-1}\varepsilon = \frac{\ell_e}{\lambda}$ determine the extent of the perturbation of the plasma by the probe (for instance the extent along B is of order of $(\gamma^{-1}\varepsilon)^{-1}R$) they do not affect the collected current. The basic reason why does not appear in I^e is that, while the collecting fields become weaker in the z_2-layer as increases, the extension of the perturbation is larger so that transverse diffusion fills the "shadow" of the probe over longer distances. Moreover, if the weak dependence on n_0^e were retained in (3.), both and would enter the expression for I^e.

The multiple scales method of Bogolinbov was used.(*) It allowed us to demonstrate the structure of the space around the probe. In two interior z-layers collisions were unimportant, while in z_2 they were dominant, except for a subregion around ($\zeta = 0, \xi_k < 1$) where a transition between both situations occurs. This sublayer could not be separated from the z_2-layer: its characteristic length changed continuously of order of magnitude. Moreover, only for $\xi_k \leq 1$ is present.

The existence of an overshooting of the potential inside the "shadow", which goes to infinity as $\sigma \to 0$, simplified the complication of this sublayer because the repelling potential field and the Maxwellian character of the electrons in the rest of the z_2-space allow to state that the electrons are Maxwellian in the whole z_2-layer.

As α increases beyond values of order $\ell n\, \sigma^{-1}$ successive points on the probe (from $\xi_k = 1$ to $\xi_k = 0$) reach a monotonic potential from the probe to infinity.

The accuracy of the solution depends, first, on the kinetic equations used and the transport coefficients obtained from them. The corrected Balescu-Lenard equations have a high degree of accuracy for ε small;

(*) This method appears suitable for problems such as collisionless shock waves.

however transport coefficients are not available for all ranges of parameters of interest, not even for a pure B-L model. As used in the present study errors of order $(\ln\Lambda)^{-1}$ are expected, although if $\ell_e < \lambda_D$ this is not generally true.

Second, the accuracy of the expansion itself depends on the small parameters. Because one cannot exclude the possibility of logarithms or fractional powers of these parameters, an analysis of the following terms in the expansion is necessary to determine the errors.

Finally, the numerical computations are accurate to 1%.

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Appendix A

We want to find the equation satisfied by a system of non-interacting particles in the presence of electric and magnetic fields. The Lagrangian for such a system is (using cylindrical coordinates).

$$L = L(q_k, \mathring{q}_k) = T - QV + \frac{Q}{c}\vec{A} \cdot \vec{W} \quad (A.1)$$

Here $q_1 = r$, $q_2 = \theta$, $q_3 = z$; Q and M are the charge and the mass of a particle; $\vec{A}$ and V are the vector and scalar potentials of the field. We have for $\vec{B}$ and $\vec{E}$

$$\vec{B} = \frac{\partial}{\partial \vec{r}} \times \vec{A} = \frac{1}{r}\begin{vmatrix} \vec{1}_r & \vec{1}_\theta & \vec{1}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}; \quad \vec{E} = - \frac{\partial V}{\partial \vec{r}}$$

Hamilton's equations are

$$\frac{\partial H}{\partial p_k} = \mathring{q}_k, \quad \frac{\partial H}{\partial q_k} = -\mathring{p}_k$$

and the Liouville equation for a cloud of one-particle systems in this six-dimensional phase space is

$$\frac{\partial F_1}{\partial t} + \sum_k \frac{\partial H}{\partial p_k} \frac{\partial F_1}{\partial q_k} - \sum_k \frac{\partial H}{\partial q_k} \frac{\partial F_1}{\partial p_k} = 0$$

where

$$\dot{p}_k = \frac{\partial L}{\partial \dot{q}_k}, \quad \frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k} \qquad \text{(A.2)}$$

Then (A.2) becomes

$$\frac{\partial F_1}{\partial t} + \left(\dot{r}\frac{\partial}{\partial r} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{z}\frac{\partial}{\partial z}\right)F_1 + \left\langle\left\{Mr\,\dot{\theta}^2 - Q\frac{\partial V}{\partial r}\right.\right.$$

$$\left. + \frac{Q}{c}\left[\frac{\partial A_r}{\partial r}\dot{r} + \frac{\partial A_\theta}{\partial r} r\dot{\theta} + A_\theta\dot{\theta} + \frac{\partial A_z}{\partial r}\dot{z}\right]\right\}\frac{\partial}{\partial p_r}$$

$$+ \left\{-Q\frac{\partial V}{\partial \theta} + \frac{Q}{c}\left[\frac{\partial A_r}{\partial \theta}\dot{r} + \frac{\partial A_\theta}{\partial \theta} r\dot{\theta} + \frac{\partial A_z}{\partial \theta}\dot{z}\right]\right\}\frac{\partial}{\partial p_\theta} +$$

$$\left. + \left\{-Q\frac{\partial V}{\partial z} + \frac{Q}{c}\left[\frac{\partial A_r}{\partial z}\dot{r} + \frac{\partial A_\theta}{\partial z} r\dot{\theta} + \frac{\partial A_z}{\partial z}\dot{z}\right]\right\}\frac{\partial}{\partial p_z}\right\rangle$$

$$\times \; F_1 = 0$$

The momenta are, from (A.1),

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\dot{r} + \frac{Q}{c}A_r$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Mr^2\dot{\theta} + \frac{Q}{c}A_\theta r$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = M\dot{z} + \frac{Q}{c}A_z$$

and therefore we can write

$$w_r \equiv \dot{r} = \frac{P_r - \frac{Q}{c} A_r}{M}$$

$$w_\theta \equiv r\dot{\theta} = \frac{P_\theta - \frac{Q}{c} A_\theta r}{Mr}$$

$$w_z \equiv \dot{z} = \frac{P_z - \frac{Q}{c} A_z}{M}$$

and

$$F(r,\theta,z,w_r,w_\theta,w_z) \equiv F(r,\theta,z,w_r\left\{P_r,r,\theta,z\right\},$$

$$w_\theta\left\{P_\theta,r,\theta,z\right\},\ w_z\left\{P_z,r,\theta,z\right\})$$

$$\equiv F_1(r,\theta,z,P_r,P_\theta,P_z).$$

Then we obtain

$$\frac{\partial F}{\partial t} + w_r \frac{\partial F}{\partial r} + \frac{w_\theta}{r} \frac{\partial F}{\partial \theta} + w_z \frac{\partial F}{\partial z} - \frac{Q}{M}\left(\frac{\partial V}{\partial r}\frac{\partial F}{\partial w_r} + \right.$$

$$\left. + \frac{\partial V}{\partial \theta}\frac{\partial F}{\partial w_\theta} + \frac{\partial V}{\partial z}\frac{\partial F}{\partial w_z}\right) + \frac{(w_\theta)^2}{r} \frac{\partial F}{\partial w_r} -$$

$$- \frac{w_\theta w_r}{r} \frac{\partial F}{\partial w_\theta} + \frac{Q}{Mc}\left(\vec{w} \times \left[\frac{\partial}{\partial \vec{r}} \times \vec{A}\right]\right) \cdot \frac{\partial F}{\partial \vec{w}} = 0$$

If $\frac{\partial}{\partial t} = 0$, $\frac{\partial}{\partial \theta} = 0$ and $\vec{B} \equiv \frac{\partial}{\partial \vec{r}} \times \vec{A} = B\vec{1}_z$
there results

$$w_r \frac{\partial F}{\partial r} + w_z \frac{\partial F}{\partial z} - \frac{Q}{M}\left(\frac{\partial V}{\partial r} \frac{\partial F}{\partial w_r} + \frac{\partial V}{\partial z}\frac{\partial F}{\partial w_z}\right) +$$

$$+ \frac{w_\theta}{r}\left(w_\theta \frac{\partial F}{\partial w_r} - w_r \frac{\partial F}{\partial w_\theta}\right) + \frac{QB}{Mc}\left(w_\theta \frac{\partial F}{\partial w_r} -\right.$$

$$\left. - w_r \frac{\partial F}{\partial w_\theta}\right) = 0 \qquad (A.3)$$

Eq. (A.3) is the left-hand side of (1.4) and (1.5) for s = 1 ($\xi \equiv r$, $\eta \equiv \theta$)

Appendix B

The Fokker Planck collision operator is given in any of the following equivalent forms(*):

$$\left(\frac{\delta F^a}{\delta t}\right)_{F-P} \equiv \frac{4\pi e^4 \ln\Lambda Z_a^2}{m_a^2} \sum_b Z_b^2 \left(-\frac{\partial}{\partial \vec{w}} F^a \cdot \frac{\partial}{\partial \vec{w}} H^{ab} + \frac{1}{2} \frac{\partial^2}{\partial \vec{w} \partial \vec{w}} F^a \right.$$

$$\left. : \frac{\partial^2}{\partial \vec{w} \partial \vec{w}} G^{ab} \right) \qquad (B.1)$$

$$\left(\frac{\delta F^a}{\delta t}\right)_{F-P} \equiv \frac{4\pi e^4 \ln\Lambda Z_a^2}{m_a^2} \sum_b Z_b^2 \left(4\pi \frac{m_a}{m_b} F^a F^b + \frac{m^b - m^a}{m^b + m^a} \right.$$

$$\left. \times \frac{\partial H^{ab}}{\partial \vec{w}} \cdot \frac{\partial F^a}{\partial \vec{w}} + \frac{1}{2} \frac{\partial^2 G^{ab}}{\partial \vec{w} \partial \vec{w}} : \frac{\partial^2 F^a}{\partial \vec{w} \partial \vec{w}} \right) \qquad (B.2)$$

$$\left(\frac{\delta F}{\delta t}\right)_{F-P} \equiv \frac{2e^4}{m_a^2} Z_a^2 \sum_b Z_b^2 \frac{\partial}{\partial \vec{w}} \cdot \int d\vec{w}' \int d\vec{k}\, \delta(\vec{k} \cdot [\vec{w} - \vec{w}'])\frac{\vec{k}\vec{k}}{k^2}$$

$$\cdot \left(F^b \frac{\partial F^a}{\partial \vec{w}} - \frac{m_a}{m_b} F^a \frac{\partial F^b}{\partial \vec{w}'} \right) \qquad (B.3)$$

where

$$H^{ab} = \frac{m_b + m_a}{m_b} \int \frac{F^b d\vec{w}'}{|\vec{w} - \vec{w}'|} \; ; \quad G^{ab} = \int F^b \, |\vec{w} - \vec{w}'| \; d\vec{w}' \qquad (B.4)$$

(*) See for instance I. P. Shkarofsky, T. W. Johnston and M. P. Bachynsky, "The Particle Kinetics of Plasmas," Addison-Wesley, Read., Mass., 1966, Chapter 7.

are the Rosenbluth potentials [56].

The Balescu-Lenard collision operator is very similar to the third form given above for $\left(\frac{\delta F}{\delta t}\right)_{F-P}$ (*):

$$\left(\frac{\delta F}{\delta t}\right)_{B-L} \equiv \frac{2e^4}{m_a^2} Z_a^2 \sum_b Z_b^2 \frac{\partial}{\partial \vec{w}} \cdot \int d\vec{w}' \int d\vec{k}\ \delta(\vec{k} \cdot [\vec{w} - \vec{w}'])\frac{\vec{k}\vec{k}}{|\varepsilon^+|^2}$$

$$\cdot\left(F^b \frac{\partial F^a}{\partial \vec{w}} - F^a \frac{\partial F^b}{\partial \vec{w}'} \frac{m_a}{m_b}\right) \qquad (B.5)$$

where ε^+ is the contracted product of $\vec{k}\vec{k}$ and the dielectric tensor.

In (B.3) the integral is logarithmically divergent at k equal to zero and infinity. Two cut-offs have to be introduced: $k_{max} = 3\lambda_L^{-1}$ and $k_{min} = \lambda_D^{-1}$. They were first introduced by Chandrasekhar [33]. k_{max} is ill-defined and often is chosen, simply, λ_L^{-1}; λ_L is the classical distance of closest approach, $\frac{e^2}{kT}$. In [33], the inter-particle distance was chosen for k_{min}^{-1} ($k_{min} = N^{1/3}$). Spitzer (see [34]) proved that the Debye length is a more proper choice for k_{min}^{-1}. The logarithmic dependence of $\frac{\delta F}{\delta t}$ on these parameters and the large value of the ratio $\frac{k_{max}}{k_{min}}$ for a classical plasma make the result insensitive to some variation in this ratio. λ_D represents the collective screening: $\lambda_D = \left(\frac{kT_e}{4\pi Ne^2}\right)^{1/2}$.

If the ions, with charge $+Z_i e$, contribute to the screening (in slow processes), then λ_D should be more properly written

$$\lambda_D = \left(\frac{kT_e}{4\pi e^2 N \left(1 + \frac{T_e}{T_i} Z_i\right)} \right)^{1/2}$$

In (B.5), the integral is divergent at $k = \infty$ and again $k_{max} \approx 3\lambda_L^{-1}$ has to be introduced; but the factor $|\varepsilon^+|^{-2}$ includes the (dynamic) screening at small k.

The mean free path for the coulombian collisions represented by (B.1) - (B.6) can be found approximately by a simple dimensional argument. A magnitude with dimension of length and inversely proportional to the density is

$$\lambda = \frac{b'(kT_e)^2}{Ne^4} = \frac{b'}{N\lambda_L^2} \qquad \text{(B.7)}$$

From elementary kinetic theory this corresponds to a hard-sphere type of interaction, with interaction radius of order λ_L. Because collisions are not strictly binary, however, b' is not a purely numerical factor but depends on the density:

$$b' = \frac{b}{\ln\Lambda}\ , \qquad b \approx (2\pi)^{-1}$$

One can also obtain equation (B.7) from the first equation of the B-B-G-K-Y hierarchy.

It is possible to speak of a mean free path (m.f.p.) for different processes; it is apparent that interaction of electrons with each other, momentum interchange between ions, electrons, and interaction between the ions themselves are characterized by the same m.f.p. (The relaxation times will differ because of different average velocities).

However, for energy interchange between electrons and ions, $b' \sim \left(\frac{m_e}{m_e}\right)^{1/2}$. Thus the energy coupling is relatively weak as compared to other aspects of the interaction. This can be shown from (B.3) or (B.5). In both integrals there appears the quantity

$$F^b \frac{\partial F^a}{\partial \vec{w}} - F^a \frac{\partial F^b}{\partial \vec{w}'} \frac{m_a}{m_b} \tag{B.8}$$

If a means electrons and b means ions, the second term is of order $\left(\frac{m_e}{m_i}\right)^{1/2}$ as compared to the first. We can expand $\frac{\delta F}{\delta t}$ in terms of this ratio. We drop the second term in (B.8) and write

$$\delta(\vec{k}\cdot[\vec{w}-\vec{w}']) \approx \delta(\vec{k}\cdot\vec{w}) \qquad (B.9)$$

and drop the summation over the ions in $\mathcal{E}^+$. Then for both the F-P and the B-L operators the integration over $\vec{w}'$ is immediate; the result is

$$\frac{\delta F^e}{\delta t} = N_i R(F^e) + O\left(\left[\frac{m_e}{m_i}\right]^{1/2}\right) \qquad (B.10)$$

Eqs. (B.9) and (B.10) are used throughout Chapter III.

Before showing the cause of the long m.f.p. for energy interchange, we demonstrate that $R(F^e)$ vanishes whenever the operator R acts on an isotropic function. To see this observe that both (B.3) and (B.5) vanish for F^a and F^b Maxwellian at the same temperature, because then

$$F^b \frac{\partial F^a}{\partial \vec{w}} - F^a \frac{\partial F^b}{\partial \vec{w}'} \frac{m_a}{m_b} = \frac{m_a}{kT} F^b F^a (\vec{w} - \vec{w}')$$

Because this is multiplies by $\vec{k}$ and there is a factor $\delta(\vec{k}\cdot[\vec{w}-\vec{w}'])$ in the integration over $\vec{k}$ (or over $\vec{w}'$), the integral vanishes.(*)

(*) The B-L operator with the magnetic field included satisfies the same property; see [42].

Now if F^e is isotropic

$$\frac{\partial F^e}{\partial \vec{w}} = \frac{\vec{w}}{w} \frac{\partial F^e}{\partial w} \qquad (B.11)$$

Dropping the second term in (B.8) and writing (B.9) implies that the product

$$\delta(\vec{k} \cdot \vec{w})\vec{k} \cdot \vec{w} \; \frac{1}{w} \; \frac{\partial F^e}{\partial w}$$

appears in an integration in R, if (B.11) is satisfied. Thereby $R(F^e) = 0$.

A similar argument causes the large m.f.p. cited above. Integrating (B.3) or (B.5) over $\frac{\vec{w}^2}{2} d\vec{w}$ and using (B.10) we have

$$\int \frac{\delta F^e}{\delta t} \; \frac{\vec{w}^2}{2} \; d\vec{w} = \frac{N_i}{2} \int R(F^e)\vec{w}^2 d\vec{w} + O\left(\left[\frac{m_e}{m_i}\right]^{1/2}\right) \qquad (B.12)$$

$R(F^e)$ is of the type (see (B.3) or (B.5) and (B.8), (B.9) and (B.10)):

$$R(F^e) = \frac{\partial}{\partial \vec{w}} \cdot \int \vec{k} \; d\vec{k} \; A(\vec{w}, F^e) \; \delta(\vec{k} \cdot \vec{w})$$

Then in (B.12) any integral such as

$$\int w_x^2 d\vec{w} \frac{\partial}{\partial w_y} \int k_y \; d\vec{k} \; A(\vec{w}, \; F^e) \; \delta(\vec{k} \cdot \vec{w})$$

vanishes because the integration over w_y can be done and $F^e(w_y \pm \infty) = 0$. Those integrals such as

$$\int w_x^2 \, d\vec{w} \frac{\partial}{\partial w_x} \int k_x d\vec{k} \; A(\vec{w}, F^e) \; \delta(\vec{k} \cdot \vec{w})$$

can be summed to yield

$$- 2 \int d\vec{w} \int d\vec{k} \; \vec{w} \cdot \vec{k} \; A(\vec{w}, F^e) \; \delta(\vec{k} \cdot \vec{w})$$

and again this integral vanishes because of the product

$$(\vec{w} \cdot \vec{k}) \; \delta(\vec{w} \cdot \vec{k})$$

The result is that only the term $O\left(\left[\frac{m_e}{m_i}\right]^{1/2}\right)$ in (B.10) contributes to the interchange of energy between ions and electrons.

It was observed in Section I-C that the orders of magnitude of the m.f.p.'s were not changed by the presence of a large magnetic field as long as $\lambda_L \ll \ell_e$ (*). A comparison was made by Rosenberg and Wu [57], by numerical calculations, for the F-P and B-L operators and the relaxation times proved to be practically unchanged.

(*) We are talking of relaxation processes in velocity space; in physical space, the transport processes are significantly modified.

Finally we point out another interesting property of the B-L operators. Assume that the system is near equilibrium so that F^a, F^b can be linearized around a Maxwellian distribution. One has to substitute $F_M^a + \Delta F^a$ and $F_M^b + \Delta F^b$ for F^a and F^b; ΔF^a and ΔF^b are small corrections to the Maxwellian distributions. Then retaining terms of first order in $|\mathcal{E}^+|^{-2}$ gives a zero contribution because F_M^a and F_M^b appear in (B.8) and the collision operator vanishes. Thus the special dependence of the collision operator on F^a, F^b (the dynamic screening) disappears, and the F-P and B-L terms are extremely alike.

Appendix C

We consider here typical ranges of magnitude of the five lengths present in this problem, which can be achieved in a thermally ionized cesium plasma of the Q-machine type.

The lengths are given by the expressions

$$\ell_e = \frac{m_e c}{eB}\left(\frac{kT_e}{m_e}\right)^{1/2} \sim T_e^{1/2} \quad B^{-1}$$

$$\ell_i = \frac{m_i c}{z_i eB}\left(\frac{kT_i}{m_i}\right)^{1/2} \sim T_i^{1/2} \quad m_i^{1/2} \quad z_i^{-1} B^{-1}$$

$$\lambda_D = \left(\frac{kT_e}{4\pi N \, e^2}\right)^{1/2} \sim T_e^{1/2} \quad N_\infty^{-1/2}$$

$$\lambda = \frac{(kT_e)^2}{2\pi N_\infty e^4 \ln \Lambda} \sim T_e^2 N_\infty^{-1}\left[\ln T_e^{3/2} \quad N_\infty^{-1/2} + \text{constant}\right]$$

$$R \sim R$$

The four non-dimensional length ratios used in this analysis are

$$\sigma \equiv \frac{l_e}{R} \sim T_e^{1/2} B^{-1} R^{-1}$$

$$\gamma \equiv \frac{\lambda_D}{l_e} \sim N_\infty^{-1/2} B$$

For a typical thermal Cs plasma, we have the following conditions

$$T_e \approx T_i \approx 2{,}300^\circ k$$

$$z_i = 1$$

$$\left(\frac{m_e}{m_i}\right)^{1/2} = 0.0020$$

$$10^{10} < N_\infty < 10^{13} \ (\text{cm.}^{-3})$$

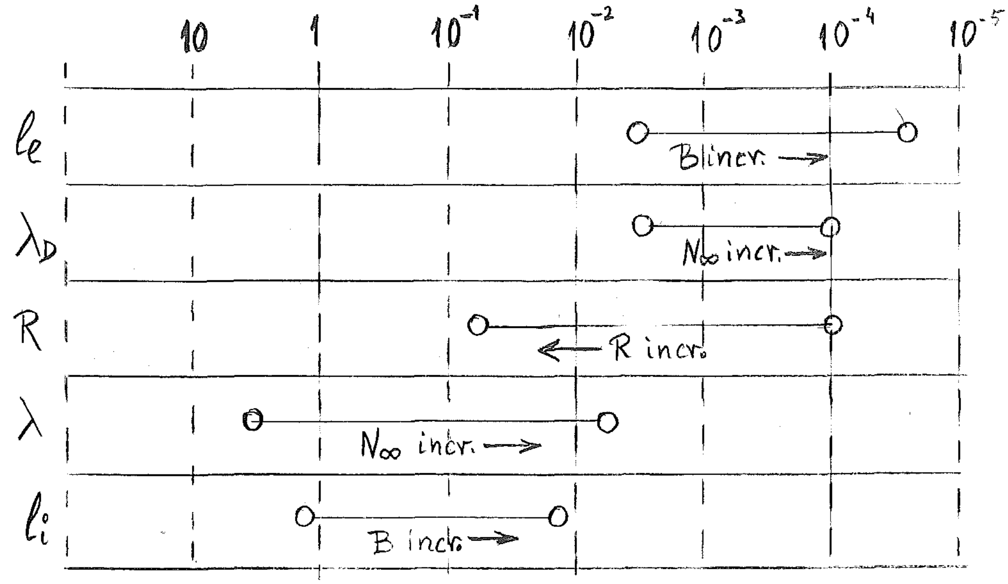
$$6.09 < l_n \Lambda < 9.54$$

Typical values of R and B are

$$10^{-4} < R < 5 \times 10^{-2} \ (\text{cm.})$$

$$3 \times 10^2 < B < 3 \times 10^4 \quad (\text{Gauss})$$

The extreme ranges of the five characteristic lengths are illustrated below



As a typical case, we consider $R = 10^{-2}$ cm., $T = 2,3 \times 10^{2\,\circ}$k, $B = 10^4$ Gauss. Then

$$\sigma = 0.01; \ \mu = 0.002; \quad \frac{l_i}{R} = \frac{\sigma}{\mu} = 5$$

For $N_\infty = 10^{12}$ cm.$^{-3}$ and the above values for T and B

$$\begin{aligned} \mathcal{E} &= 0.007 \\ \gamma &= 3.16 \end{aligned} \tag{C.1}$$

In section I-B, it was shown that the ratio $\frac{B'}{B}$ of the perturbation field to the external field is of order

$$\frac{B'}{B} = 0\left(\frac{kT_e}{m_e c^2} \ \frac{l_e \lambda}{\lambda_D^2}\right) = 0\left(\frac{kT_e}{m_e c^2} \ (\gamma \mathcal{E})^{-1}\right)$$

and

$$\frac{kT_e}{m_e c^2} \approx 1.7 \times 10^{-5} \qquad \text{for} \quad T_e = 10^5\ {}^{o}k$$

Because $(\gamma \varepsilon)^{-1}$ grows with T_e as $T_e^{3/2}$ and is independent of N_∞, (except for the weak logarithmic variation) for $B = 10^4$ Gauss and $T_e = 10^5\ {}^{o}k$ we have from (C.1)

$$\frac{B'}{B} = 0 \left(1.7 \times 10^{-5} \times \frac{1}{0.007 \times 3.16} \times \left[\frac{10^5}{2.3 \times 10^3} \right]^{3/2} \right)$$

$$= 0(10^{-1})$$

Thus $T_e \approx 10^5$ can be considered a critical temperature, in this problem, as far as the neglection of $\frac{B'}{B}$ is concerned; $\frac{B'}{B}$ decreases very fast as T_e decreases.

Appendix D

In 1963 Frieman and Book [29] found a differential equation for the correlation function of an electron plasma in equilibrium, $\Phi(r)$; they demonstrated its validity for all distances r between particles to zero order in $\frac{\lambda_L}{\lambda_D} = (3N_D)^{-1}$, while the Debye-Huckel correlation, $\frac{-r}{\lambda_L} e^{-r/\lambda_D}$, is valid only for $r \ll \lambda_L$. The equation was

$$\frac{d^2\Phi}{dr^2} + \left(\frac{2}{r} - \frac{\lambda_L}{r^2}\right) \frac{d\Phi}{dr} - \frac{\Phi}{\lambda_D^2} = 0$$

$$\Phi(0) = -1 \qquad (D.1)$$

$$\Phi(\infty) = 0$$

They used the "inner and outer" expansion method to solve this equation for $\frac{r}{\lambda_D} = O(1)$ and $\frac{r}{\lambda_L} = O(1)$ and matched in an intermediate region $\frac{r}{N^{-1/3}} = O(1)$. Unfortunately the final solution

$$\Phi = -1 + e^{-\lambda_L/r} + \frac{\lambda_L}{r} - \frac{\lambda_L}{r} e^{-r/\lambda_D} + O\left(\left[\frac{\lambda_L}{\lambda_D}\right]^{2/3}\right)$$

gives a divergent equation of state. In 1964 Shure found that a function Φ^* existed such that

$$\frac{\Phi^*}{\Phi} = 1 + 0\left(\frac{\lambda_L}{\lambda_D}\right) \text{ uniformly in } r.$$

He found $\Phi^* = (e^{-\lambda_L/r} - 1)e^{-r/\lambda_D}$ which gives a convergent equation of state.

We shall now derive Φ^* directly using the multiple scales method. As pointed out by Frieman [50], Bogoliubov had both the mathematical method and the physical insight to develop the theory which Frieman [50] and Sandri [51] introduced in 1963. It is also curious that Frieman did not use his multiple scales method to solve (D.1).

In effect it is tempting to write directly from (D.1)

$$\Phi = \Phi\left(\frac{r}{\lambda_L}, \frac{r}{\lambda_D}, \frac{\lambda_L}{\lambda_D}\right) = \Phi(r_0, \mathcal{E}r_0, \mathcal{E})$$

Here $\mathcal{E} = \dfrac{\lambda_L}{\lambda_D}$ and <u>not</u> $\dfrac{\lambda_D}{\lambda}$; also $r_0 = \dfrac{r}{\lambda_L}$, $r_1 = \mathcal{E}r_0 = \dfrac{r}{\lambda_D}$.

Upon expanding then Φ and $\frac{d}{dr}$ we find

$$\frac{1}{\lambda_L^2}\left[\left(\frac{\partial}{\partial r_0} + \varepsilon\frac{\partial}{\partial r_1} + \dots\right)^2 + \left(\frac{2}{r_0} - \frac{1}{r_0^2}\right) \times \right.$$

$$\left. \times \left(\frac{\partial}{\partial r_0} + \varepsilon\frac{\partial}{\partial r_1} + \dots\right) - \varepsilon^2\right]$$

$$\times \left[\Phi_0 + \varepsilon\Phi_1 + \dots\right] = 0 \qquad \text{(D.2)}$$

$$\Phi_0(0) = -1, \ \Phi_0(\infty) = 0$$

$$\Phi_j(0) = \Phi_j(\infty) = 0 \qquad j \geqslant 1 \qquad \text{(D.3)}$$

Then we can solve the equations of successive order in ε:

$$\frac{\partial^2 \Phi_0}{\partial r_0^2} + \left(\frac{2}{r_0} - \frac{1}{r_0^2}\right)\frac{\partial \Phi_0}{\partial r_0} = 0$$

or

$$\Phi_0 = A_1(r_1, \ldots) + A_2(r_1, \ldots)\, e^{-1/r_0}$$

(D.4)

Using (D.4) in the following equation we have

$$\frac{\partial^2 \Phi_1}{\partial r_0^2} + \left(\frac{2}{r_0} - \frac{1}{r_0^2}\right) \frac{\partial \Phi_1}{\partial r_0} + 2\, \frac{\partial A_2}{\partial r_1}\, \frac{e^{-1/r_0}}{r_0^2} +$$

$$+ \left(\frac{2}{r_0} - \frac{1}{r_0^2}\right) \left(\frac{\partial A_2}{\partial r_1}\, e^{-1/r_0} + \frac{\partial A_1}{\partial r_1}\right) = 0$$

(D.5)

Using the method suggested in Section II-C

$$\lim_{r_j \to \infty} \frac{\partial}{\partial r_j} = 0 \quad ,$$

let us multiply (D.5) by r_0 and let $r_0 \to \infty$; then we get

$$\frac{\partial A_2}{\partial r_1} + \frac{\partial A_1}{\partial r_1} = 0$$

or

$$A_1 = - A_2 + C(r_2, \ldots)$$

$$\Phi_0 = C(r_2, \ldots) + A_2(r_1, \ldots)(e^{-1/r_0} - 1)$$

(D.6)

The following equation (in ε^2) is

$$\left(2 \frac{\partial^2}{\partial r_0 \partial r_2} + \frac{\partial^2}{\partial r_1^2} - 1\right) \Phi_0 + \frac{\partial^2 \Phi_2}{\partial r_0^2} + 2 \frac{\partial \Phi_1}{\partial r_0 \partial r_1} +$$

$$+ \left(\frac{2}{r_0} - \frac{1}{r_0^2}\right) \left(\frac{\partial \Phi_2}{\partial r_0} + \frac{\partial \Phi_1}{\partial r_1} + \frac{\partial \Phi_0}{\partial r_2}\right) = 0$$

(D.7)

Because all the terms in (D.1) have appeared already it is expected that with this equation the description of Φ_0 will be complete.

Letting $r_0 \rightarrow \infty$ in (D.7), there results

$$C = 0$$

having used (D.6) for Φ_0 in (D.7).

Multiplying by r_0 and letting $r_0 \rightarrow \infty$, (D.7) becomes

$$\frac{\partial^2 A_2}{\partial r_1^2} - A_2 = 0 \qquad \text{(D.8)}$$

provided: a) $r_0 \dfrac{\partial^2 \Phi_2}{\partial r_0^2} \rightarrow 0$ as $r_0 \rightarrow \infty$, which is correct if $\dfrac{\partial \Phi_2}{\partial r_0} \rightarrow 0$ as $r_0 \rightarrow \infty$; and b) $\Phi_1 \rightarrow 0$ as $r_0 \rightarrow \infty$. Condition b) does not follow from our method but condition a) does. As we commented at the end of Chapter II, this method is often incomplete.

That $\Phi_1 \rightarrow 0$ as $r_0 \rightarrow \infty$ does not follow from the second condition

$$\Phi_j(\infty) = 0 \qquad j \geqslant 1$$

in (D.3) (because this infinity is in the r_1 scale) but from observing that $\Phi_0(r_0 \rightarrow \infty) \rightarrow 0$ (see (D.6)) and elimination then of secularities

$$\lim_{r_k \to \infty} \frac{\Phi_j}{\Phi_{j-1}} = O(1)$$

The solution therefore of (D.8) gives

$$A_2 = B_1 e^{r_1} + B_2 e^{-r_1} \tag{D.9}$$

Using then condition $\Phi_0(\infty) = 0$ and $\Phi_0(0) = -1$ in (D.6), with (D.9) we get

$$\Phi_0 = \left(e^{-1/r_0} - 1\right) e^{-r_1} \equiv \left(e^{-\lambda_L/r} - 1\right) e^{-r/\lambda_D}$$

Appendix E

It was observed in Appendix B that the B-L equation gives results very similar to those obtained from the F-P equation. While the similarity of the equations suggests that this should be so, several authors have recently used the B-L model for numerical computation of transport coefficients and confirmed this result.

In the absence of a magnetic field, Spitzer and Harm [52] computed the electric conductivity from the F-P equation. The result was given in the form (σ <u>in this Appendix only</u> is the conductivity).

$$\sigma = \gamma_E(z_i)\,\sigma_L \tag{E.1}$$

where

$$\sigma_L = \frac{2}{\pi^{3/2}} \frac{(2kT)^{3/2}}{m_e^{1/2}\, e^2 \ln\Lambda\, z_i} \tag{E.1'}$$

is the conductivity of a Lorentz plasma. The function $\gamma_E(z_i)$ was given numerically: it varies between $\gamma_E(1) = 0.582$ and $\gamma_E(\infty) = 1$. A very good agreement existed with the value obtained by means of the Chapman-Enskog method in the Boltzmann equation, if terms through the fourth approximation are retained [58]. Edward and

Sanderson [59] using the Green's function method to solve Liouville's equation, found $\gamma_E(1) = 0.561$, which is about 3% smaller than Spitzer's result. Braun [60] solved the B-L equation and found a correction factor to (E.1) of the form

$$\left[1 + 0.245\ (\ln\Lambda)^{-1}\right]^{-1}$$

giving for a typical $\ln\Lambda = O(10)$, a value about 2.5% smaller than Spitzer's value.

Itikawa [61] used a B-L corrected equation (following Aono and Kihara's formulation [31]) and found

$$\gamma_E(1) = 0.574\ \frac{1 + \frac{7.299}{21.237}(\ln\Lambda)^{-1} - O\ (\ln\Lambda)^{-2}}{1 + \frac{11.046}{10.892}(\ln\Lambda)^{-1} - O\ (\ln\Lambda)^{-2}}$$

He gave the corrections in $(\ln\Lambda)^{-2}$. This expression gives about 4% less than Spitzer's formula for $\ln\Lambda = O(10)$.

Other transport coefficients are given in [60], [61] and [31]. For instance, for the thermal conductivity, Braun gave a correction factor

$$\left[1 + 0.228\ (\ln\Lambda)^{-1}\right]^{-1}$$

to Spitzer's result. Sundaresan and Wu [62] and Rand and Levinsky [63] obtained, for specific cases, correction factors for the thermal conductivity of about 1.06 and 0.97.

In general a slight decrease appears to exist from the value given by the F-P equation for both a B-L and a B-L corrected equation.

We shall use these results for the determination of the fluxes $n_1^e u_z^{e1}$ and $n_2^e u_\xi^{e2}$ used in Chapter III. While $n_1^e u_z^{e1}$ is obtained from an integral equation, $n_2^e u_\xi^{e2}$ reduces to an integration; it will be performed here to illustrate a frequent cause for the similar results from the F-P and the B-L equations.

The equation for the electric conductivity is (see (3.55))

$$\frac{e}{m_e} \frac{\partial V}{\partial z} \frac{\partial F_0^e}{\partial w_z} = \left(\frac{\delta F^e}{\delta t}\right)_1$$

If B does not enter the collision process, we have

$$(N_e \overline{w}_z^e)_1 = \frac{\sigma}{e} \frac{\partial V}{\partial z}$$

Taking into account the factor $\frac{\beta + 1}{\beta}$ in (3.55), (E.1) and (E.1') and using the Braun correction there results

$$(N_e \overline{w}_z^e)_1 = \frac{2(2kT_e)^{3/2}\, \gamma_E(z_1)}{\pi^{3/2} m_e^{1/2} z_1 e^3\, \ell n \Lambda}\left(1 + \frac{0.245}{\ell n \Lambda}\right)^{-1} \frac{\beta + 1}{\beta} \frac{\partial V}{\partial z}$$

where $\overline{w}_z^e = v_z^e\, U_e$.

Changing to non-dimensional variables, we obtain finally

$$n_1^e u_z^{e1} = (N_\infty \lambda_L^2 \lambda \, \ell n \Lambda)^{-1} \left[1 + 0.245 \, (\ell n \Lambda)^{-1}\right]^{-1} \frac{\gamma_E}{z_i} \, x$$

$$x \; \frac{2^{5/2}}{\pi^{3/2}} \; \frac{\beta + 1}{\beta} \; \alpha \; \frac{\partial \phi_0}{\partial z_2} \qquad (E.2)$$

This equation is (3.56).

For $n_2^e u_\xi^{e2}$ we had (see (3.46b)):

$$n_2^e u_\xi^{e2} = \frac{\beta + 1}{(2\pi)^{3/2}} \frac{\partial n_0^e}{\partial \xi_k} \int v_\eta \, d\vec{v} \left[\frac{\delta}{\delta t} \left\{ f_0^e, v_\eta \, e^{-\frac{v2}{2}} \right\}_L + \right.$$

$$\left. + \frac{\delta}{\delta t} \left\{ v_\eta \, e^{-v^2/2}, \, f_0^e \right\}_L + n_0^1 R_L (v_\eta \, e^{-v^2/2} \;) \right] \qquad (E.3)$$

The first two terms balance each other if collisions are local (from the point of view of both inhomogeneities and external fields) because electrons do not gain average momentum from themselves. If B enters the collision process, they are not local; however, it has been shown [64] that diffusion arising from like particles is a higher-order effect unless the gradient scales are comparable to the Larmor radius. Thus

$$n_2^e u_\xi^{e2} = \frac{\beta + 1}{(2\pi)^{3/2}} n_0^e \frac{\partial n_0^?}{\partial \xi_K} \int v_\eta \, d\vec{v} \, R_L(v_\eta \, e^{-v^2/2})$$

When $\lambda_D < l_e$, R_L for the F-P equation takes the form (from (2.1), (2.11), (B.2) and the definition of R in Appendix B):

$$R_L(v_\eta \, e^{-v^2/2}) = \frac{\lambda N_\infty}{U_e^4} \frac{4\pi e^4 \ln\Lambda}{m_e^2} z_i \left[\left(\frac{\partial}{\partial \vec{v}} \frac{1}{v}\right) \cdot \frac{\partial}{\partial \vec{v}} v_\eta \, e^{-v^2/2} + \frac{1}{2}\left(\frac{\partial^2}{\partial \vec{v}\,\partial \vec{v}} v\right) : \frac{\partial^2}{\partial \vec{v}\,\partial \vec{v}} v_\eta \, e^{-v^2/2}\right] \qquad (E.4)$$

Using the identities

$$v_j \frac{\partial}{\partial v_j} = \vec{v} \cdot \frac{\partial}{\partial \vec{v}} = v \frac{\partial}{\partial v}$$

$$\left(\frac{\delta_{ij}}{v} - \frac{v_i v_j}{v^3}\right) \frac{\partial^2}{\partial v_i \, \partial v_j} = \frac{1}{v}\left(\frac{\partial}{\partial \vec{v}} \cdot \frac{\partial}{\partial \vec{v}} - \frac{\partial^2}{\partial v^2}\right)$$

and expressing the Laplacian $\frac{\partial}{\partial \vec{v}} \cdot \frac{\partial}{\partial \vec{v}}$ in spherical velocity coordinates ($v_z = v\nu$, $v_\xi = v \sin(\cos^{-1}\nu)\cos\omega$, $v_\eta = v \sin(\cos^{-1}\nu) \times \sin\omega$) the bracket in (E.4) becomes

$$\frac{1}{2v^3}\left[\frac{\partial}{\partial \nu}(1 - \nu^2)\frac{\partial}{\partial \nu} + \frac{1}{1-\nu^2}\frac{\partial^2}{\partial \omega^2}\right] v \sin(\cos^{-1}\nu) \times \sin\omega \, e^{-v^2/2}$$

$$= \frac{e^{-v^2/2}}{2v^3} v \sin(\cos^{-1}\nu) \sin\omega \frac{\cos 2(\cos^{-1}\nu) - 1}{1 - \nu^2}$$

$$= - v_\gamma \frac{e^{-v^2/2}}{v^3}$$

Thereby we have

$$n_2^e u_\xi^{e2} = \frac{(\beta + 1) n_0^e}{(2\pi)^{3/2}} \frac{\partial n_0^e}{\partial \xi_k} 4\pi N_\infty \lambda_L^2 \lambda \ln\Lambda z_i$$
$$\times \int v_\gamma^2 \frac{e^{-v^2/2}}{v^3} d\vec{v} =$$

$$= - \left(\frac{2}{\pi}\right)^{1/2} \frac{4\pi}{3} (\beta + 1) z_i (N_\infty \lambda_L^2 \lambda \ln\Lambda) n_0^e \frac{\partial n_0^e}{\partial \xi_k}$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{4\pi}{3} z_i (N_\infty \lambda_L^2 \lambda \ln\Lambda) \frac{\beta + 1}{\beta} e^{-\frac{2\alpha\phi_0}{\beta}} \tag{E.5}$$
$$\times \left(- \frac{\partial \phi_0}{\partial \xi_k} \right)$$

Using now the B-L equation, we have (from (2.1), (2.11) and (B.5)[(*)]

(*) According to what we said at the end of Appendix B, the distribution function appearing in $\mathcal{E}^+$ is Maxwellian when the collision term is linearized around equilibrium.

$$R_L(v_\eta\, e^{-V^2/2}) = \frac{\lambda N_\infty}{U_e^4}\,\frac{2e^4 z_i}{m_e^2}\,\frac{\partial}{\partial v_j}\int d\vec{k}\,\frac{k_j k_\ell\,\delta(\vec{k}\cdot\vec{v})}{|\varepsilon^+|^2}$$

$$\times\, v_\eta\, e^{-V^2/2}$$

To perform the integration over $v_\eta\, d\vec{v}$ we must evaluate the integral

$$E = \int v_\eta d\vec{v}\,\frac{\partial}{\partial v_j}\int d\vec{k}\,\frac{k_j k_\ell\,\delta(\vec{k}\cdot\vec{v})}{|\varepsilon^+|^2}\,\frac{\partial}{\partial v_\ell}\, v_\eta\, e^{-V^2/2}$$

Integrating first with respect to $\vec{v}$ by parts we get

$$E = -\int d\vec{v}\int d\vec{k}\;\frac{k_\eta k_\ell\,\delta\left(\frac{\vec{k}}{k}\cdot\vec{v}\right)}{k\,|\varepsilon^+|^2}\left(-\,v_\eta\,v_\ell\, e^{-V^2/2} + \delta_{\ell\eta}\, e^{-V^2/2}\right)$$

$$= -\int d\vec{v}\int d\vec{k}\;\frac{\delta\left(\frac{\vec{k}}{k}\cdot\vec{v}\right)}{k|\varepsilon^+|^2}\left(-\,k_\eta v_\eta\;\vec{k}\cdot\vec{v}\, e^{-V^2/2} + k_\eta^2\, e^{-V^2/2}\right)$$

The first term within the bracket does not contribute to the integral because of the product

$$\frac{\vec{k}}{k}\cdot\vec{v}\,\delta\left(\frac{\vec{k}}{k}\cdot\vec{v}\right)$$

In the remaining integral we integrate again with respect to $\vec{v}$. For a Maxwellian distribution

$$\mathcal{E}^+ = k^2 + k_D^2 \left[1 - 2\,\frac{\vec{k}}{k}\cdot\vec{v}\; e^{-\left(\frac{\vec{k}}{k}\cdot\vec{v}\right)^2} \operatorname{erf}\left(\frac{\vec{k}}{k}\cdot\vec{v}\right) + \right.$$

$$\left. + \,i\,\pi^{1/2}\,\frac{\vec{k}\cdot\vec{v}}{k}\; e^{-\left(\frac{\vec{k}}{k}\cdot\vec{v}\right)^2} \right]$$

so that if $\frac{\vec{k}\cdot\vec{v}}{k} = 0$, $\mathcal{E}^+ = k^2 + k_D^2$ (*). Then to integratė over $\vec{v}$ we choose a spherical velocity coordinates system with axis of azimuths parallel to $\vec{k}$; we have to do the integral

$$\int_0^{2\pi} d\omega_k \int_{-1}^{+1} d\nu_k \int_0^{\infty} v^2 dv \; \frac{e^{-v^2/2}\,\delta(v\nu_k)}{\left|\mathcal{E}^+(v\nu_k)\right|^2} =$$

$$= 2\pi \int_0^{\infty} v^2 dv \, \frac{e^{-v^2/2}}{\left|\mathcal{E}^+(0)\right|^2} = \frac{2\pi}{(k^2+k_D^2)^2}$$

Therefore

$$E = -\int \frac{d\vec{k}}{k}\, k_\eta^2 \frac{2\pi}{(k^2+k_D^2)^2} = \frac{-2\pi}{3}\, 4\pi \int_0^{3k_L} \frac{k^3\, dk}{(k^2+k_D^2)^2}$$

We have cut the integral at $k_{max} = \left(\frac{\lambda_L}{3}\right)^{-1}$. Finally,

---

(*) See "Statistical Mechanics of Charged Particles," R. Balescu, Interscience, New York, 1963, Chapter III.

$$E = -\frac{4\pi^2}{3}\left[\ln\{(\Lambda)^2 + 1\} + \frac{1}{\Lambda^2 + 1} - 1\right]$$

$$= -\frac{8\pi^2}{3}\left[\ln\Lambda\,(1 - \frac{0.5}{\ln\Lambda}) + O(\Lambda^{-2})\right]$$

Thus we obtain

$$(n_2^e u_\xi^{e2})_{B-L} = \left(\frac{2}{\pi}\right)^{1/2} \frac{4\pi}{3}\, z_1 (N_\infty \lambda_L^2 \lambda\, \ln\Lambda)\, \frac{\beta + 1}{\beta}\, \alpha\, e^{-\frac{2\alpha\phi_0}{\beta}}$$

$$\times \frac{\partial\Phi_0}{\partial\xi_k}\,(1 - \frac{0.5}{\ln\Lambda}) = (n_2^e u_\xi^{e2})_{F-P} \times (1 - \frac{0.5}{\ln\Lambda})$$

For $\ln\Lambda = O(10)$ the difference term $\left[1 - 0.5(\ln\Lambda)^{-1}\right]$ is about 5%.

The arbitrariness in k_{max} should also give errors of this order of magnitude. Using a B-L corrected equation due to Kihara and Aono (see the footnote on page 15), Itikawa, Kihara and Aono [31] obtained for the transverse diffusion coefficient (for $T_e = T_i$).

$$D_\perp = \frac{8\pi}{3}\left(\frac{c}{B}\right)^2 N_\infty e^2 z_1 \left(\frac{m_e}{2\pi k T_e}\right)^{1/2}$$

$$\times \ln\Lambda\left[1 - \left(\ln \frac{3\gamma^2 z_1}{4}\,(z_1 + 1)^{\frac{z_1 + 1}{2z_1}}\right)(\ln\Lambda)^{-1}\right]$$

(E.6)

where $\gamma = 0.78$ ... is Euler's constant; the appearance of z_1 is equivalent to the use of a refined λ_D. Eq.

(E.5) in dimensional form differs from (E.6) only in the factor multiplying $(\ln\Lambda)^{-1}$ in the bracket of (E.6).

Finally we should consider the variation appearing in these coefficients from the presence of the magnetic field if it affects the collision process. Belyaev [36] and Gurevich and Firsov [65] give for $D_\perp$, when $\ell_e \ll \lambda_D$

$$D_\perp = \frac{8\pi}{3}\left(\frac{c}{B}\right)^2 N_\infty e^2\left(\frac{m_e}{2\pi kT_e}\right)^{1/2}\left[\ln\frac{\ell_e}{\lambda_L} + \right.$$

$$\left. + \frac{3}{2}\ln\left(\frac{m_i}{m_e}\right)^{1/2}\ln\frac{\lambda_D}{\ell_e}\right] \qquad \text{(E.7)}$$

(In [65] the thermal DeBroglie wavelength

$$\lambda = \frac{\hbar}{(2m_e kT_e)^{1/2}}$$

appeared instead of λ_L; their approach is quantum-mechanical, necessary if $T_e > 4\times10^5\ {}^oK$).

It may be noted that $D_\perp$ is not obtained by simple substitution of ℓ_e for λ_D in Λ, but there is an additional term which can be much larger than the first.

Golant [66], Aliev and Shister [67] and Silin and Shister [68] calculate $D_\perp$ for different combinations of inequalities between $\lambda_L \frac{m_i T_e}{m_e T_i}$, λ_D, ℓ_e and ℓ_i; the first length seems to be relevant while λ_L is not. The case of interest here, when $\lambda_D > \ell_e$, is

$$\ell_e < \lambda_D < \ell_i < \lambda_L \frac{m_i}{m_e} \frac{T_e}{T_i}$$

The above authors give for this orderin; $\left(\text{if } \frac{T_i}{T_e} = O(1)\right)$

$$\ell n \frac{\ell_e}{\lambda_L} + \frac{3}{4} \ell n \frac{(\lambda_D \ell_e)^{1/2}}{\lambda_L} \ell n \frac{\lambda_D}{\ell_e}$$

for the square bracket in (E.7).

For the conductivity along $\vec{B}$ no results are available for the case where $\vec{B}$ enters the collision operator.

Appendix F

F1. The build-up of the potential

It was found in Chapter III that $n_{e_0} \approx n_{i_0} \ll 1$ and $\chi_0 \gg 1$ as $z_1 \to \infty$. At $z_0 = 0$, $\chi_0 = O(1)$. It is of interest to know if the potential is built up in the z_0 or in the z_1-layer; i.e., to know if $\chi_0 = O(1)$ or $\chi_0 \gg 1$ as $z_0 \to \infty$. If the last were the case, χ_0 could actually decrease from $z_1 = 0$ to $z_1 \to \infty$, or a potential well for electrons could exist near the probe. In both cases n_0^e could be $O(1)$ somewhere in the z_1-layer and then in (3.66b) the collision term would be not negligible. Moreover, if a well existed, the trapping effect on the ions could affect possibly the collection of the electrons.

To study the field in z_0, let us consider Poisson's equation in this layer, (3.24)

$$\frac{\partial^2 \chi_0}{\partial z_0^2} = n_0^e - n_0^i \tag{F.1}$$

Let us assume for the moment that both ions and electrons are Maxwellian in z_0. Then

$$\frac{\partial^2 \chi_0}{\partial z_0^2} = ae^{\chi_0} - be^{-\chi_0} \tag{F.2}$$

where

$$a = n_0^e(z_0 = 0)e^{-\chi_p}$$
$$b = n_0^i(z_0 = 0)e^{\chi_p} \qquad (F.3)$$

A first integral of (F.1) is

$$\left(\frac{\partial \chi_0}{\partial z_0}\right)^2 = 2\, ae^{\chi_0} + 2\, be^{-\chi_0} + 2c \qquad (F.4)$$

where c is a constant (in z_0) of integration. Writing $e^{\chi_0/2} = q$ we obtain

$$a^{1/2}\, 2^{-1/2} z_0 = \int_{q_p}^{q} \frac{dq}{\left(q^4 + \frac{c}{a} q^2 + \frac{b}{a}\right)^{1/2}} \qquad (F.5)$$

This is an elliptic integral of the first kind. The manner of reducing it to a canonical form depends on the roots of the denominator in the integrand. Both a and b are obviously positive. It is possible to show that $c < 0$, $\left(\frac{c}{a}\right)^2 - 4\frac{b}{a} \geqslant 0$ and that the correct transformation should be

$$\frac{q}{q_1} = \sin \tau \qquad (F.6)$$

where

$$q_1^2 = \frac{-c - (c^2 - 4ab)^{1/2}}{2a}$$
$$q_2^2 = \frac{-c + (c^2 - 4ab)^{1/2}}{2a} \qquad (F.7)$$

are the roots of the polynominal in (F.5). If we define

$$k^2 \equiv \left(\frac{q_1}{q_2}\right)^2 < 1$$

(F.5) becomes

$$a^{1/2}\, 2^{-1/2}\, z_0 = (q_2)^{-1} \int_{\tau_p}^{\tau} \frac{d\tau}{(1-k^2 \sin^2\tau)^{1/2}} \equiv (q_2)^{-1}\left[F(k,\tau) - F(k,\tau_p)\right] \tag{F.8}$$

where F is the elliptic integral of the first kind of modulus k.

From (F.8), $F(k,\tau) \to \infty$ as $z_0 \to \infty$. Only for $k = 1$ is F unbounded; therefore $q_1 = q_2$ and from (F.7)

$$c = -2(ab)^{1/2}$$

Use of this expression in (F.4) imposes a certain slope at $z_0 = 0$ in function of the densities at the probe.

Moreover $F(1,\tau) \to \infty$ only for $\tau \to \frac{\pi}{2}$. Thereby from (F.6)

$$\lim_{z_0 \to \infty} \sigma = q_1 \equiv \left(\frac{-c}{2a}\right)^{1/2} = \left(\frac{b}{a}\right)^{1/4} = \left(\frac{m_0^i}{m_0^e}\bigg|_{z=0} e^{2\chi_p}\right)^{1/4} = \lim_{z_0 \to \infty} e^{\chi_0/2} \tag{F.9}$$

Thus if the ratio of densities at the probe is large enough, $\chi_0(z_0 \to \infty)$ is large and calculable according to (F.9).

It is possible to verify in (F.2) and (F.4) that the value of χ_0 given in (F.9) satisfies both

$$\lim_{z_0 \to \infty} n_0^e = \lim_{z_0 \to \infty} n_0^i$$

$$\lim_{z_0 \to \infty} \frac{\partial \chi_0}{\partial z_0} = 0$$

The function $F(1,\tau)$ has a simple form

$$F(1,\tau) = \frac{1}{2} \ln \frac{1 + \sin \tau}{1 - \sin \tau}$$

Thus from (F.8) we obtain

$$\chi_0 = \frac{1}{2} \ln \frac{b}{a} + 2 \ln \left[e^{\chi_p/2} \left(\frac{a}{b}\right)^{1/4} + \frac{1 - e^{-(ab)^{1/4} z_0}}{1 + e^{-(ab)^{1/4} z_0}} \right]$$
$$- 2 \ln \left[1 + e^{\chi_p/2} \left(\frac{a}{b}\right)^{1/4} \frac{1 - e^{-(ab)^{1/4} z_0}}{1 + e^{-(ab)^{1/4} z_0}} \right] \qquad \text{(F.10)}$$

Then we find

$$\left.\frac{\partial \chi_0}{\partial z_0}\right|_{z_0=0} = 2^{1/2} \left[b^{1/2} e^{-\chi_p/2} - a^{1/2} e^{\chi_p/2} \right] \qquad \text{(F.11)}$$

Now for $\xi_k > 1$, $\left.\frac{\partial \chi_0}{\partial z_0}\right|_{z_0=0} = 0$ by symmetry.

Therefore

$$e^{\frac{\chi_0(z_0=0)}{2}} = \left(\frac{b}{a}\right)^{1/4} \qquad \text{(F.12)}$$

But this is also the value at $z_0 \to \infty$ (see (F.9)). In fact, by using (F.12) in (F.10), there results

$$\chi_0 = \frac{1}{2} \ln \frac{b}{a} = \chi_0 (z_0 = 0)$$

This means that for $\xi_K > 1$ there is no build up of the potential.

The assumption of both ions and electrons being Maxwellian in z_0 is acceptable for $\xi_K > 1$ (the electrons certainly are; the ions would be if the build up were in z_0). The above result for $\xi_K > 1$ is, qualitatively at least, valid. However for $\xi_K < 1$ the Maxwellian assumption is strongly incorrect because the probe is absorbing; the ions, which are attracted(*) are indeed very far from being Maxwellian.

If the ions are assumed Maxwellian at $z_0 \to \infty$, with the adequate cut-off to take into account the absorption on the probe, it can be found that no build-up is possible. This result does not depend on the functional form of f_0^i as $z_0 \to \infty$ but on the "absorptiveness" of the probe and the attraction of ions to it; then the conservation of flux in z_0 requires

$$\frac{\partial}{\partial z_0} n_0^i u_z^{i0} = 0$$

(*) We are considering values of χ_p to the left of the value for which the overshooting disappears.

and the fact that all ions travel toward the probe in z_0 does not allow n_0^i to grow as $z_0 \to 0$ so as to build up the field.

The fact, then, that n_0^i cannot grow as required from $z_0 \to \infty$ to $z_0 = 0$ (essentially because $\lambda_D \ll R$ and thus there is conservation of z-flux in z_0) eliminates the possibility of χ_0 being large as $z_0 \to \infty$. In the following z-scale, there is a ξ-flux of ions (but not of electrons because they are inhibited in their transverse motion) and this allows the increase in n_0^i to build up the potential. The extension of the Debye layer where quasineutrality has not to be satisfied has been discussed in Section III-F.

F2. The ξ_j-gradients ($j < K$)

It was stated in Chapter III that as $z_1 \to \infty$ the strong gradients around $\xi_K = 1$ have been smoothed. The basic reason is that, because of these large gradients, the ξ-flux is very large there and its variation with ξ is also large because the ξ-flux is not large at either $\xi_j \to -\infty$ or $\xi_j \to +\infty$ ($j < K$). Then a large variation in z-flux is possible through the continuity equation. The sudden jump in z-flux at $z_0 = 0$ is allowed therefore to decrease, as z increases, much faster than that for other values of ξ_K.

The exact extent to which this occurs is difficult to determine unless a simultaneous analysis is made of the ξ_K-region in z_1 for the ions. However, it should be pointed out that in (3.39), where $\frac{\partial f_2^e}{\partial \omega}$ appears and which allows the determination of $n_2^e u_\xi^{e2}$, the limit $z_1 \rightarrow \infty$ had been taken. In the z_1-layer there is an additional term

$$\delta_1^z \left\{ v_z \frac{\partial}{\partial z_1} + \frac{\partial \chi_0}{\partial z_1} \frac{\partial}{\partial v_z} \right\} \Delta_1^e f_1^e$$

Because $\Delta_1^e f_1^e = -\sigma \sin\omega\, D_\xi\, f_0^e + \Delta_{12}^e f_{12}^e$ it is noted that this term gives a u_ξ^{e2}-contribution. In effect (3.40) would be of the form

$$\Delta_2^e \frac{\partial f_2^e}{\partial \omega} = A(\omega) + \delta_1^z \sigma \left\{ v_z \frac{\partial}{\partial z_1} + \frac{\partial \chi_0}{\partial z_1} \frac{\partial}{\partial v_z} \right\} \sin\omega\, D_\xi f_0^e$$

where A is the right-hand side of (3.40). The complete form for $n_2^e u_\xi^{e2}$ is $\Delta_2^e n_2^e u_\xi^{e2} = -\int A v_\nu\, d\vec{v} - (\delta_1^z \sigma) \int v_\perp \left\{ v_z \frac{\partial}{\partial z_1} + \right.$ $\left. + \frac{\partial \chi_0}{\partial z_1} \frac{\partial}{\partial v_z} \right\} D_\xi\, f_0^e \quad \sin^2\omega\ d\vec{v}$ and the second term is different from zero. The order of magnitude of this ξ-flux is $\sigma \delta_1^z = \frac{\ell_e^2}{\lambda R}$, the same as that produced by collisions (see (3.46a)), but this only in the ξ_K-region. In a ξ_j-region σ should be changed into $\delta_j^\xi = \frac{\ell_e}{L_j^\xi}$ if $j < K$. Then the continuity equation would require

$$\frac{1}{L_j^{\xi}}\left[\delta_j^{\xi}\left(\frac{l_e}{\lambda}\right)\right]=\left(\frac{1}{\lambda}\right)\sigma$$

or

$$\delta_j^{\xi}=\sigma^{+1/2}$$
$$L_j^{\xi}=\sigma^{-1/2}\,l_e \gg l_e$$

For l_e not much smaller than λ_D, the gradients in the λ_D and l_e scales disappear as $z_1 \to \infty$. An analysis of the ξ_j-region, with $j < K$ can be made for z_2 and agreement is found with this result.

ERRATA (Sanmartin)

Page	
3	Figure title should read "... for $B \approx 0$".
4	Line 11, replace "in" with "on".
6	Line 16, reads "...(see [2] and ...)".
10	Line 21, correct spelling is "anisotropy".
12	Line 16, delete "such".
13	Last line, clarify subscript on $N_\infty^{1/2}$.
14	First line, delete "it".
	Line 19, write "perfectly".
18	Delete the three improperly erased lines at the end of the page.
22	Line 17, write "collisions".
32	Fourth line from bottom, remove left-hand parenthesis.
33	Line 14, to be moved to left-hand margin.
37	Line 3, replace "Fig. 6" with "Fig. 5".
	Footnote, bracket reference [2].
38	Line 18, write "Bogoliubov".
	Lines 19 and 20, write "non-linear mechanics" and "celestial mechanics".
53	Eq. (3.14), insert "+" before " $\frac{1}{\xi^s} \sum_j$ ...".
	Line 2 of text, inequality is "$P_1 \neq P_2$".
65	Line 5 from bottom, set "where" on line.
66	Line 14 of footnote, write "Bohm".

ERRATA (Sanmartin)

67 Line 1, write "$z_1 \to \infty$;".

68 Line 11, write "n_0^e".

70 Last equation is Eq. (3.43).

77 Line 16, insert comma following "I^1".

Line 20, insert comma following "(s = 1)".

84 Line 13, insert "by" following "understood".

85 Footnote, insert comma following "any f_{12}^e".

92 Line 11, write "$\xi_K^e < 1$".

Line 15, write "changes".

93 Line 5, write "The".

113 Line 8, write "For j = 0, $\Theta_0 = 0$."

115 Line 11, write "$\psi^{(m-1)}$".

119 Lines 22, 23, write "Z_1" for "z_1".

127 Insert the following equation at the bottom of the page:

$$\sigma\left(\frac{\partial\psi}{\partial\xi}\right) \text{ is finite}$$

129 Line 9, write "intervene". Line 16, write "as λ"

Line 15, write "... why λ does ...".

Line 20, write "... retained in (3.58), both $\frac{\lambda_D}{\lambda}$ and $\frac{\ell_e}{\lambda}$ would ...".

Line 22, write "Bogoliubov".

130 Line 8, write "... only for $\xi_K < 1$ is it present".

134 Ref. , write " . N. Bogoliubov".

ERRATA (Sanmartin)

141 Line 3, delete "(*)".

144 Line 14, write "multiplied".

148 At bottom of page, insert the following equations

$$\epsilon = \frac{\lambda_D}{\lambda} \sim T_e^{-3/2} N_\infty^{1/2} [\ln T_e^{3/2} N_\infty^{-1/2} + \text{const.}]$$

$$\mu \equiv \frac{l_e}{l_i} \sim Z_i^{1/2} (m_e T_i / m_i T_e)^{1/2}$$

149 Last line, write "... are illustrated on the following page."

150 Line 2 of text, write "$T = 2.3 \times 10^2$ °K".

155 The unclear exponent in Eq. (D.5) is $(- 1/r_0)$.

159 Next to last line, write Enskog.

160 Line 14, write "... given in [60]".

Line 15, write "... [61] and [31]".

Line 17, formula should read $[1 + \ldots]^{-1}$

176 The following equation should appear at the bottom of p. 176:

$$\frac{1}{L_j^\xi} \left[\delta_j^\xi \left(\frac{l_e}{\lambda} \right) \right] = \left(\frac{1}{\lambda} \right) \sigma$$

177 First equation, change exponent to read

$$\delta_j^\xi = \sigma^{+1/2}$$

ERRATA (Sanmartin)

Replace "k" with "K" in the text, in the inequality "$j < k$", or as a subscript on ξ_k, as follows: Page (line) = 56(5), 75(6,7,9,13,16,19), 80(7), 81(5), 83(2,8, 9,14,14,16,16,19,20), 84(7), 87(last), 92(11), 116(17,17,17, 18,18,18), 117(4,5,5,6,7,9,13), 119(5,6,6,8,18), 130(4,8, 16), 174(2,5,7,8,), 175(14,16,20), 176(1,4,19,21), 177(6).